Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference

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Motivation and Introduction

Motivation

- SVARs are widely used for policy analysis but...
  - No workable rank conditions to ascertain global identification
  - No efficient algorithms for small-sample estimation and inference when nonlinear identifying restrictions are used.

Main Contributions of the Study

1. General rank conditions for global identification of both identified and exactly identified models.
2. Easy application of these conditions and implementable into many identifying restrictions (linear + some non linear)
3. Rank condition for exactly identified models is a straightforward counting exercise
4. Develop efficient algorithms for small sample estimation and inference.

Identifying Restrictions

$$a_{11}\Delta \log P_{c,t} + a_{31}R_t = c_1 + b_{11}\Delta \log P_{c,t-1} + b_{21}\Delta \log Y_{t-1} + b_{31}R_{t-1} + \varepsilon_{1,t}$$
$$a_{12}\Delta \log P_{c,t} + a_{22}\Delta \log Y_t = c_2 + b_{12}\Delta \log P_{c,t-1} + b_{22}\Delta \log Y_{t-1} + b_{32}R_{t-1} + \varepsilon_{2,t}$$
$$a_{13}\Delta \log P_{c,t} + a_{23}\Delta \log Y_t + a_{33}R_t = c_3 + b_{13}\Delta \log P_{c,t-1} + b_{23}\Delta \log Y_{t-1} + b_{33}R_{t-1} + \varepsilon_{3,t}$$

- Restriction 1: R_t responds sluggishly to output but can respond to commodity prices
- Restriction 2: MP has no long-run effect on output (neutrality)
- Restriction 3: Output responds sluggishly to changes in R_t but can respond to commodity prices

Alternative World
Consider a model in which a shock in commodity markets has no long-run effect on output, but MP has a long run effect on output. We still have 3 restrictions, but which model is *globally identified*?

Rothenberg (1971)

- Order condition (simply counting restrictions). Identified if $n(n-1)/2$ restrictions exist.
- Only necessary condition! Need to show global identification!
- Linear restrictions on the covariance matrix of shocks imply nonlinear restrictions on the structural parameters.
- Checking whether an SVAR is globally identified $\equiv$ Checking whether a system of nonlinear restrictions on the structural parameters has a unique solution!

Global Identification

Set up of the SVAR

$$\mathbf{y}_t'\mathbf{A}_0 = \sum_{l=1}^{p} \mathbf{y}_{t-l}'\mathbf{A}_l + \mathbf{c} + \varepsilon_t' \quad \text{for } 1 \leq t \leq T$$

Define

$$\mathbf{A}_+' \equiv \begin{pmatrix} \mathbf{A}_1' & \dots & \mathbf{A}_p' & \mathbf{c}' \end{pmatrix} \text{ and } \mathbf{x}_t' \equiv \begin{pmatrix} \mathbf{y}_{t-1}' & \dots & \mathbf{y}_{t-p}' & 1 \end{pmatrix}$$

$$\mathbf{y}_t'\mathbf{A}_0 = \mathbf{x}_t'\mathbf{A}_+ + \varepsilon_t'$$

Reduced form is

$$\mathbf{y}_t' = \mathbf{x}_t'\mathbf{B} + \mathbf{u}_t' \qquad \mathbf{B} = \mathbf{A}_+\mathbf{A}_0^{-1}, \mathbf{u}_t' = \varepsilon_t'\mathbf{A}_0^{-1}$$

Parameters of the reduced form are therefore $(\mathbf{B}, \Sigma)$

Rothenberg (1971) Identification

Two parameter points $(\mathbf{A}_0, \mathbf{A}_+)$ and $(\tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_+)$ are **observationally equivalent** if and only if they imply the same distribution of $\mathbf{y}_t$ for $1 \leq t \leq T$. (i.e., they have the same reduced form representation $(\mathbf{B}, \mathbf{\Sigma})$).

- These two parameter points have the same reduced form representation if and only if there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{A}_0 = \tilde{\mathbf{A}}_0\mathbf{P}$ and $\mathbf{A}_+ = \tilde{\mathbf{A}}_+\mathbf{P}$
- $(\mathbf{A}_0, \mathbf{A}_+)$ is *globally identified* iff there is no other parameter point that is observationally equivalent.
  Because observational equivalence is the same as finding $\mathbf{P}$, the set of all $n \times n$ orthogonal matrices $(O(n))$ is crucial.

Rank Condition for Global Identification

For $1 \leq j \leq n$ and any $k \times n$ matrix $\mathbf{X}$, define $\mathbf{M}_j(\mathbf{X})$ as

$$\mathbf{M}_j(\mathbf{X}) = \begin{pmatrix} \mathbf{Q}_j\mathbf{X} \\ \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix} \end{pmatrix}$$

Theorem 1
Consider an SVAR with admissible restrictions R. If $(\mathbf{A}_0, \mathbf{A}_+) \in R$ and $\mathbf{M}_j(f(\mathbf{A}_0, \mathbf{A}_+))$ is of rank n for $1 \leq j \leq n$, then the SVAR is globally identified at $(\mathbf{A}_0, \mathbf{A}_+)$.

Theorem 2 (for Partial Identification)
Consider an SVAR with admissible restrictions R. If $(\mathbf{A}_0, \mathbf{A}_+) \in R$ and $\mathbf{M}_j(f(\mathbf{A}_0, \mathbf{A}_+))$ is of rank n for $1 \leq j \leq n$, then the j-th equation is globally identified at the parameter point $(\mathbf{A}_0, \mathbf{A}_+)$. Back to IRFs

Under Exact Identification

- Consider an SVAR with admissible restrictions R. The SVAR is exactly identified iff, for almost any reduced-form parameter point $(\mathbf{B}, \Sigma)$, there exists a unique structural parameter point $(\mathbf{A}_0, \mathbf{A}_+) \in R$ such that $g(\mathbf{A}_0, \mathbf{A}_+) = (\mathbf{B}, \Sigma)$
- Equivalently, an SVAR with restrictions R is exactly identified iff, for almost every structural parameter point $(\mathbf{A}_0, \mathbf{A}_+) \in U$, there $\exists$ a unique matrix $\mathbf{P} \in O(n)$ such that $(\mathbf{A}_0\mathbf{P}, \mathbf{A}_+\mathbf{P}) \in R$.
- Theorem 6: The SVAR is exactly identified iff the total number of restrictions is equal to $n(n-1)/2$ and the rank condition in theorem 1 is satisfied for some $(\mathbf{A}_0\mathbf{P}, \mathbf{A}_+\mathbf{P}) \in R$
- Theorem 7: However, consider an SVAR with admissible and strongly regular restrictions represented by R. The SVAR is exactly identified iff $q_j = n - j$ for $1 \leq j \leq n$.

Back to IRFs

Local vs. Global Identification

Consider a three-variable SVAR

$$a_{11}y_{1t} + a_{21}y_{2t} = \varepsilon_{1t}$$
$$a_{22}y_{2t} + a_{32}y_{3t} = \varepsilon_{2t}$$
$$a_{13}y_{1t} + a_{33}y_{3t} = \varepsilon_{3t}$$
$$\mathbf{A}_0 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

- We have 1 restriction in each equation (3 in total), so $q_1 = q_2 = q_3 = 1$.
- Model satisfies the Rothenberg condition $(n = 3) \implies 3(3-1)/2 = 3$
- Model is locally identified at the parameter point $a_{11} = a_{22} = a_{33} = 1$ and $a_{13} = a_{21} = a_{32} = 2$.
- But based on Theorem 7, it is not identified at the parameter point

Local vs. Global Identification

- Why is it not identified at that point? Consider

$$P = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{pmatrix}$$

  Easy to show that

$$\tilde{A}_0 = A_0 P = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

- **Hence, $\tilde{A}_0$ satisfies the restrictions and is observationally equivalent to A_0.** Therefore, a structural model can be locally identified but nonetheless fails to be globally identified.

Applications

Monetary SVAR

- Typically, monetary SVARs impose restrictions on $(\mathbf{A}_0, \mathbf{A}_+)$ based on economic interpretations of the parameters. Typically separate MP equation from money demand equation and other non-policy equations.
- Consider a 5 variable model, $\log Y$, $\log P$, R, $\log M$, and $\log P_c$.

$$\mathbf{A}_0 = \begin{array}{c} \\ \log Y \\ \log P \\ R \\ \log M \\ \log P_c \end{array} \begin{array}{c} \begin{array}{ccccc} \text{PS} & \text{PS} & \text{MP} & \text{MD} & \text{Inf} \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ 0 & a_{22} & 0 & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix} \end{array},$$

Figure 1: A_0

- MP (Monetary Policy), Inf (Commodity Information), MD (Money Demand), PS (Production Sector)

Monetary SVAR

- We have $k = n = 5$. To use Theorem 1, we need to build the restriction matrices $\mathbf{Q}_j$ for $j = 1, 2, 3, 4, 5$ as

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{Q}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{Q}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 2: $\mathbf{Q}_j$

- Clearly, we see that $q_1 = 4$, $q_2 = 3$, $q_3 = 3$, $q_4 = 1$ and $q_5 = 0$ which is 11 (and is greater than $n(n-1)/2 = 10$).
- By Rothenberg's (1971) order condition, the model *may* be identified.

- But since it is only a necessary condition, we apply the sufficient condition of Theorem 1 by filling the rank matrices $\mathrm{M}_j(f(\mathrm{A}_0, \mathrm{A}_+))$ for $j = 1, 2, 3, 4, 5$.

$$
\mathbf{M}_1 = \begin{bmatrix} 0 & a_{22} & 0 & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad
\mathbf{M}_2 = \begin{bmatrix} 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\mathbf{M}_3 = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ 0 & a_{22} & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & a_{55} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad
\mathbf{M}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{55} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad
\mathbf{M}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Figure 3: M_j

- Clearly, there $\exists$ values a_{ij} such that the rank of the matrix is 5 for $j = 1, .., 5$. For example, if $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{14} = 1$. Hence, by Thm 1 and 3, the model is globally identified for almost all structural parameters.

Restrictions on Impulse Responses

- Consider a four-variable quarterly SVAR with *three* contemporaneous and *three* long-run restrictions on impulse responses.
- Output growth $\Delta \log Y$, Inflation ΔP, Nominal Short-Term Interest Rate R, and Change in the Nominal Exchange Rate $\Delta \log Ex$
- **Short Run Restrictions**: MPS has no contemporaneous effect on $\Delta \log Y$, ERS has no contemporaneous effect on $\Delta \log Y$ and R
- **Long Run Restrictions**: AD shocks have no long-run effect on Y, MPS has no long run-effect on Y, ERS has no long run effect on Y

Restrictions on Impulse Responses

TABLE 1

Restrictions implying that the model is identified

$$f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{IR}_0 \\ \mathbf{IR}_\infty \end{bmatrix} =$$

	Ex	MP	D	S
$\Delta \log Y$	0	0	×	×
$\Delta \log P$	×	×	×	×
R	0	×	×	×
$\Delta \log Ex$	×	×	×	×
$\Delta \log Y$	0	0	0	×
$\Delta \log P$	×	×	×	×
R	×	×	×	×
$\Delta \log Ex$	×	×	×	×

TABLE 2

Restrictions implying that the model is only partially identified

$$f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{IR}_0 \\ \mathbf{IR}_\infty \end{bmatrix} =$$

	Ex	MP	D	S
$\Delta \log Y$	0	0	×	×
$\Delta \log P$	×	×	×	0
R	0	×	×	×
$\Delta \log Ex$	×	×	×	×
$\Delta \log Y$	0	×	0	×
$\Delta \log P$	×	×	×	×
R	×	×	×	×
$\Delta \log Ex$	×	×	×	×

Figure 4: Restrictions on the Impulse Responses

Back to IRFs

Restrictions on Impulse Responses

- In Table 1, we have that $n = 4$, $k = 2n$, $q_1 = 3$, $q_2 = 2$, $q_3 = 1$ and $q_4 = 0$. Therefore, the total number of restrictions is 6 and is equal to $n(n-1)/2 = 6$ and Rothenberg's (1971) order condition for exact identification holds. Because $q_j = n - j$ for $j = 1, 2, 3, 4$, this model is exactly identified by Thm 7. Theorem 7
- In Table 2, consider an alternate specification in which supply shocks have no contemporaneous effect on inflation (price stickiness) and *MPS* may have a long-run effect on output (non-neutrality). Table 2
- The set of restrictions is equal to 6, but using Thm 7, the model is *not* exactly identified because $q_1 = 3$ and $q_2 = q_3 = q_4 = 1$.

Restrictions on Impulse Responses

- We try to see if the model is *partially identified*?
  - Express restriction matrices $\mathbf{Q}_j$ and the rank matrices $\mathbf{M}_j(f(\mathbf{A}_0, \mathbf{A}_+))$ for $j = 1, 2, 3, 4$.
  - We find that $\mathbf{M}_j(f(\mathbf{A}_0, \mathbf{A}_+)) = 4$ for $j = 1, 3, 4$ at almost any parameter point but the rank of $\mathbf{M}_2(f(\mathbf{A}_0, \mathbf{A}_+)) = 3$. Therefore, the second, third, and fourth equations are *not identified*.
  - But by Thm 2, the first equation is identified. Theorem 2

Algorithms for Estimation and Small-Sample Inference

On Estimation

- Once global identification has been established, the next step is to perform *small-sample* estimation and inference.
- Existing estimation methods for SVARs (through MLE or Posterior) is inefficient, even more so for small-sample inference (because MCMC/bootstrap need expensive random samples of the structural parameters).
- Gali (1992) solve a system of nonlinear equations for every draw of the parameters at each time t.

Algorithm for Exactly Identified Models

Theorem 5
Consider an SVAR with restrictions represented by R. The SVAR is exactly identified iff, for almost every structural parameter point $(\mathbf{A}_0, \mathbf{A}_+) \in U$, there exists a unique matrix $\mathbf{P} \in O(n)$ such that $(\mathbf{A}_0\mathbf{P}, \mathbf{A}_+\mathbf{P}) \in R$

- **Big Idea**: For almost any value of $(\mathbf{A}_0, \mathbf{A}_+)$, there is some $\mathbf{P}$ such that $(\mathbf{A}_0\mathbf{P}, \mathbf{A}_+\mathbf{P})$ satisfies the identifying restrictions.
- Instead of solving a complicated system of nonlinear equations (like Gali, 1992), why not find a rotation matrix $\mathbf{P}$ in a very efficient manner.

Algorithm 1

Consider an exactly identified SVAR with admissible and strongly regular restrictions represented by R. Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of the unrestricted structural parameters.

- Step 1: Set $j = 1$
- Step 2: Form the matrix

$$\tilde{\mathbf{Q}}_j = \begin{pmatrix} \mathbf{Q}_j f(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{p}_1' \\ \vdots \\ \mathbf{p}_{j-1}' \end{pmatrix}$$

- Step 3: There $\exists$ a unit vector $\mathbf{p}_j$ such that $\tilde{\mathbf{Q}}_j \mathbf{p}_j = 0$ because $r(\mathbf{Q}_j) = n - j$ and hence $r(\tilde{\mathbf{Q}}_j) < n$. Use LU decomposition of $\tilde{\mathbf{Q}}_j$ to find the unit vector $\mathbf{p}_j$, the sign of which is consistent with the normalization rule.
- Step 4: If $j = n$ stop; otherwise, set $j = j + 1$ and repeat step 2.

In Words Justin can Understand

- For any structural parameter point (A_0P, A_+P), you get an orthogonal matrix

$$P = [p_1 \dots p_n]$$

  such that $(A_0P, A_+P) \in R$. By Thm 5, that P will be unique for almost all structural parameters.
- Algorithm 1 provides us with an orthogonal matrix P that rotates the unrestricted estimate to the estimate that satisfies the identifying restrictions.
- If the original estimate is for the reduced-form parameters, you can also use Algorithm 1 to rotate the Choleskey decomposition Σ

Showing an Example

- Consider a 3-variable SVAR (output growth, interest rate, and inflation).
- Three restrictions: (1) demand shocks have no LR effect on output, (2 and 3) MPS have neither a SR or LR effect on output

TABLE 3

Short- and long-run restrictions

$$
f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{IR}_0 \\ \mathbf{IR}_\infty \end{bmatrix} = \begin{array}{c} \\ \Delta \log Y \\ R \\ \log P \\ \Delta \log Y \\ R \\ \log P \end{array} \begin{array}{c} \begin{array}{ccc} \text{MP} & \text{D} & \text{S} \end{array} \\ \begin{bmatrix} 0 & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \end{array}
$$

Figure 5: SR and LR Restrictions

Showing an Example

- Is the system identified? $n = 3, q_1 = 2, q_2 = 1, q_3 = 0$. By Thm 7, the system is exactly identified. Therefore, for almost any value of $(\mathbf{A}_0, \mathbf{A}_+)$, there $\exists$ a unique rotation matrix $\mathbf{P}$ such that the restrictions hold.
- Let us now express the restrictions in terms of $\mathbf{Q}_j$ matrices

$$\mathbf{Q}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Q}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Q}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathbf{Q}}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } \bar{\mathbf{Q}}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Figure 6: Restriction Matrices

- Note we just delete the rows of zeros. Since all rows of $\mathbf{Q}_3$ are zero, there is no $\bar{\mathbf{Q}}_3$. Working with $\bar{\mathbf{Q}}_j$ is operationally easier than working with $\tilde{\mathbf{Q}}_j$ in Algorithm 1 as $\bar{\mathbf{Q}}_j$ will always be an $(n-1) \times n$ matrix.

Showing an Example

- Suppose the reduced-form parameters are

$$B = \begin{pmatrix} 0.5 & -1.25 & -1 \\ 0.5 & 0.25 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 1 & 0.5 & 1 \\ 0.5 & 4.25 & 2.5 \\ 1 & 2.5 & 3 \end{pmatrix}$$

- We the compute A_0 from the Choleskey decomposition of Σ^{-1} and $A_+ = BA_0$. Since $IR_0' = A_0^{-1}$ and $IR_\infty' = (A_0 - A_1)^{-1}$, we have that

$$f(A_0, A_+) = \begin{pmatrix} IR_0 \\ IR_\infty \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \tilde{Q}_1 = \bar{Q}_1 f(A_0, A_+) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Continuing the Procedure

- Find a unit length vector $\mathbf{p}_1$ such that $\tilde{\mathbf{Q}}_1\mathbf{p}_1 = 0$. Most computationally efficient is using LU decomposition, but QR decomposition is probably more convenient.
- Let $\tilde{\mathbf{Q}}_1' = \mathbf{Q}\mathbf{T}$ where $\mathbf{Q}$ is orthogonal and $\mathbf{T}$ is upper triangular. If we choose $\mathbf{p}_1$ to be the last row of $\mathbf{Q}$, then $\tilde{\mathbf{Q}}_1\mathbf{p}_1$ will be the last column of $\mathbf{T}'$, which is zero. Hence, in this example, $\mathbf{p}_1' = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$
- To obtain $\mathbf{p}_2$, we form

$$\tilde{\mathbf{Q}}_2 = \begin{pmatrix} \bar{\mathbf{Q}}_2 f(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{p}_1' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Then, take $\mathbf{p}_2$ to be the last row of the orthogonal component of the QR decomposition to obtain $\tilde{\mathbf{Q}}_2'$ to get that $\mathbf{p}_2' = \begin{pmatrix} -0.7071 & 0.7071 & 0 \end{pmatrix}$

Continuing the Procedure

- Keep going for $\mathbf{p}_3$ and then combine $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$ and you get that

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{pmatrix} = \begin{pmatrix} 0 & -0.7071 & -0.7071 \\ 0 & 0.7071 & -0.7071 \\ 1 & 0 & 0 \end{pmatrix}$$

- It is straightforward that $\mathbf{Q}_j f(\mathbf{A}_0\mathbf{P}, \mathbf{A}_+\mathbf{P})\mathbf{e}_j = 0$ for $1 \leq j \leq 3$.

On other Algorithms

- Much faster algorithm available for triangular systems
- But these algorithms don't apply to SVAR with sign restrictions because an SVAR with sign restrictions on the IRFs is *not locally identified*. Always exists a $\mathbf{P}$ arbitrarily close to an $\mathbf{I}$ that can satisfy. Instead, find a set of IRFs that satisfy the same sign restrictions. An algorithm for this is also proposed.
- Under a Bayesian context, a prior is needed. If $\mathbf{P}$ is an orthogonal matrix, then the transformed parameters must have the same prior density as the original parameters.

Final Thoughts

Conclusion

- Analyzing the SVAR to ascertain its identifiability is essential, otherwise, empirical results may be misleading.
- General rank conditions for global identification are proposed and are necessary and sufficient which can be checked using matrix-filling and rank-checking.
- No need to compute derivatives because we exploit the orthogonal structure.
- Efficient algorithms are great.