Mathematical Economics

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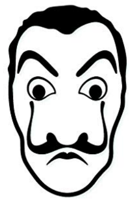
For my Dad up there

May the Force be with you, always

Preface



"In this world, everything is governed by balance. There's what you stand to gain and what you stand to lose. And when you think you've got nothing to lose, you become overconfident" - The Professor
La Casa de Papel/Money Heist

In mid 2019 right before graduation after I had secured my job search, I had bummed about just doing my hobbies and binged watch countless Netflix shows to make up for lost ground while grinding my graduate thesis. Since I would end up working for the central bank, I was recommended to watch a show called *Money Heist* by a friend (who was of course not from the central bank). The premise of the show was your typical drama/crime/thriller with a lot of well written characters, the names of which remind me of my countless travels and misadventures right before 2020 and the pandemic. I, of course, related to the professor character (being one and all) and was intrigued by the line he said above. As in anything in economics, balance is often desired. We desire this construct of equilibrium. But he posits something very game theory-ish in that when the odds seem in our favor, we can become overconfident. I guess this best describes my affinity with mathematics in the context of economics. On the one hand having gone through many courses involving math, you'd think you'd be good at it. And yet, till today, I still find myself dumbfounded by simple problems. But trust me when I say this, it is definitely okay. Do not fret too much, it is a learning process to learn math. Many may deem it a daily struggle. However, I will want to prove to you that it is an indispensable tool in the economist's tool kit, one that is often characterized as what differentiates us from other social sciences.

This is a compilation of my notes in **Mathematical Economics**. I have constructed it in such a way that it approaches the subject matter rigorously but in a more paletable and approachable manner. As this is just my personal work, there are bound to be errors to which I apologize in advance. Videos on various lessons can be found on my YouTube channel, just search Justin Eloriaga. Feel free to like, share, and subscribe as the channel also includes other courses which you might eventually end up taking. This work was primarily based on the notes of my collegues, namely, Ailyn Ang Shi and Evan Li Liao which I supplemented with a deeper discussion in some areas and more examples.

Contents

1 Differentiation

In this section we will discuss the rudiments of differentiation in economic. We will tackle the underlying basis first which means going through the concept of a derivative and how it is articulated mathematically. We then move on to the various techniques for differentiation and then move to multivariate differentiation.

1.1 Rates of Change, Derivative, and Tangent Line

1.1.1 Rates of Change

The **average rate of change** of f over the interval $[a, a+h]$, or the difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h} \tag{1}$$

In this equation, $\Delta x = h$ is called the **increment** of x, and $\Delta y = f(a+h) - f(a)$ is the respective rate of change in y. We can modify this slightly by taking the instantaneous rate of change of f at a and is represented by the limit

$$\lim_{h \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2}$$

The **derivative of a function** f is a function, denoted as f', such that its value a a number x in the domain of f is given by the form if the limit exists.

Theorem 1.1. *Provided that the limit exists, the derivative is given as* $f'(x) = \lim_{\Delta x \to 0} \frac{f(a+h) - f(a)}{h}$

To understand this concept, let us picture this scenario. Suppose a car, initially at rest, moved along a straight road in the same direction and its motion was monitored during the first 60 seconds. The following data were recorded, namely, t which is the time in seconds from the start, and s which is the distance travelled in feet from the start.

t (seconds)	15	20	25	30	35	40	45
s (feet)	168	300	468	675	918	1200	1518

Consider the time interval $15 \leq t \leq 30$. The **average speed** during this interval is

$$\frac{\Delta s}{\Delta t} = \frac{675 - 168}{30 - 15} = 33.800 \frac{\text{feet}}{\text{second}}$$

This figure which is the average speed gives us a rough idea of how fast the car is running during an interval of time. But it does not give us the speed at an instant in time. To illustrate, see the table below.

Interval Length	Average Speed Before $t = 30$	Average Speed After $t = 30$
15 seconds	$(675 - 168)/15 = 33.8$	$(1518 - 675)/15 = 56.2$
10 seconds	$(675 - 300)/10 = 37.5$	$(1200 - 675)/10 = 52.5$
5 seconds	$(675 - 468)/5 = 41.4$	$(918 - 675)/5 = 48.6$
1 second	$(675 - 630)/1 = 45.0$	$(720 - 675)/1 = 45.0$

The average speeds before and after the 30th second tend toward each other as the size of the interval becomes smaller and smaller. The common value approached by these average speeds as the interval shrinks to zero is the instantaneous speed at $t = 30$.

We say that f is a **differentiable** function and call $f'(x)$ as the **derivative of f with respect to x**. The derivative allows us to compute for the rate of change at any number within the domain of the function. We refer to equation one as the *difference quotient.*

Say we want to determine the derivative of the function $f(x) = 4x^2 + 2$. To do this, we can merely use the difference quotient.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{4(x + \Delta x)^2 + 2 - (4x^2 + 2)}{\Delta x}$$

We can simplify this as

$$f'(x) \lim_{\Delta x \to 0} \frac{4(x^2 + 2x\Delta x + \Delta x^2) + 2 - (4x^2 + 2)}{\Delta x}$$

$$f'(x) \lim_{\Delta x \to 0} \frac{4x^2 + 8x\Delta x + 4\Delta x^2 + 2 - 4x^2 - 2}{\Delta x}$$

$$f'(x) \lim_{\Delta x \to 0} \frac{8x\Delta x + 4\Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} 8x + 4\Delta x$$

We evaluate this limit and we find that

$$f'(x) = 8x$$

Find the derivatives of the following functions using the difference quotient.

1. $f(x) = 3x^2 + 1$
2. $f(x) = \frac{1}{x}, x \neq 0$

Note that our notation for differentiation may take four different forms. These are the Leibniz Notation $\frac{dy}{dx}$, the Euler notation $D_x f$, the Newton Notation $\dot{y}$, or the Lagrange Notation $f'(x)$.

1.1.2 Tangent Line

The **tangent line** to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with the slope provided that the limit below exists.

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{(a + h) - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{3}$$

Hence, the derivative is simply the **slope of the tangent to the curve** which, by definition, is the slope of the curve.

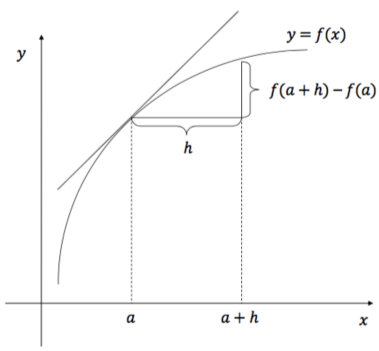


Figure 1: Slope of the Tangent Line

If we write $x = a + h$, then $h = x - a$, and h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating the slope of the tangent line is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

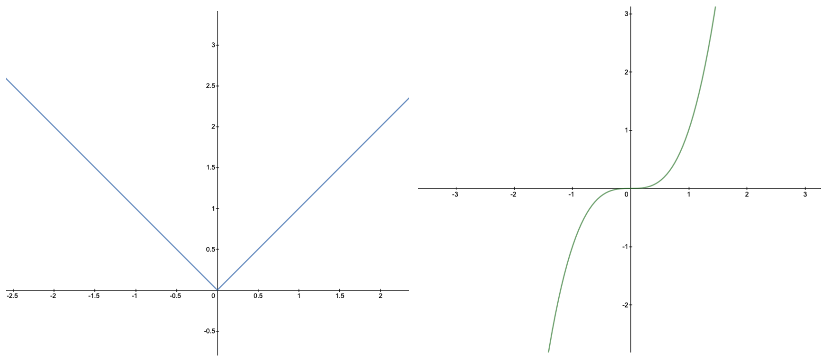
The slope of any one curvilinear function is not constant. This may differ at different points on the curve as in figure 1. A tangent line is a straight line that touches a curve at only one point. Measuring the slope of a function may require different tangent lines.

1.1.3 Differentiability and Continuity

For a function to be **differentiable** at some x^*, the tangent to the curve at x^* has to be the same tangent whether x^* is approaches from the left or from the right. That is, there should be no abrupt change in slope in the neighborhood of x^*, or in other words, the curve should be **smooth** around x^*.

Theorem 1.2. *If f is differentiable at a, then f is continous at a*

The theorem implies that differentiability implies continuity or that continuity is a necessary condition for differentiability. Note that the converse is not generally true as there are functions that are continuous but are not differentiable at certain points.



(a) $f(x) = |x|$ (b) $f(x) = x^{\frac{1}{3}}$

Figure 2: Functions Illustrating Theorem 1.2

1.2 Rules of Differentiation

We formally refer to *differentiation* as the process of finding the derivative of a function. This involves nothing more complicated than applying a few basic rules.

1.2.1 Constant Function Rule

If c is a constant, and if $f(x) = c$ for all x, then the derivative of that function is zero.

$$f'(x) = 0$$

1.2.2 Linear Function Rule

If we have a linear function $f(x) = mx + b$, the derivative of the linear function is merely m. This stems from the derivative of a variable raised to the first power is always equal to the coefficient of the variable, while the derivative of a constant is simply zero.

Say we were given with the linear function $f(x) = 7x + 49$

$$f'(x) = 7$$

1.2.3 Power Function Rule

If n is a positive integer, and if we have the function $f(x) = x^n$, then it follows that the derivative is given as

$$f'(x) = nx^{n-1}$$

For example, say we are given with $f(x) = 2x^2$. The derivative of this is simply

$$f'(x) = 2 \cdot 2x^{2-1} = 4x$$

1.2.4 Constant Multiple Rule

If f is a differentiable function, c is a constant and g is a function defined by $g(x) = cf(x)$, then it follows that

$$g'(x) = cf'(x)$$

For example, take $g(x) = 8x^2$ where c is obviously 8 and $f(x) = x^2$. We know from the Power function rule that the derivative of $f(x)$ is $f'(x) = 2x$. Hence, getting the derivative of $g(x)$ is straightforward

$$g'(x) = 8 \cdot f'(x) = 8 \cdot 2x = 16x$$

1.2.5 Sum and Difference Rules

If f and g are both differentiable functions, and if h is a function defined by $h(x) = f(x) \pm g(x)$, then the derivative of $h'(x)$ is merely the sum or difference of the derivatives of $f(x)$ which is $f'(x)$ and $g(x)$ which is $g'(x)$.

$$h'(x) = f'(x) \pm g'(x)$$

For example., say we had a function $h(x) = x^2 + 2x^5 + \frac{3}{4}x^4$. The derivative of this function $h'(x)$ is merely the derivative of each term with respect to x.

$$h'(x) = \frac{dh(x)}{dx}(x^2) + \frac{dh(x)}{dx}(2x^5) + \frac{dh(x)}{dx}\left(\frac{3}{4}x^4\right) = 2x + 10x^4 + 3x^3$$

Using the rules you know so far, take the derivative of the following functions

1. $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$
2. $f(x) = \frac{5}{x^3}$
3. $f(x) = \frac{1}{x^4} - x^4$
4. $f(x) = x^{0.5} - 2x - 1$
5. The area A of a circular disc of radius r is given by $A = \pi r^2$. Calculate the derivative of A with respect to r and give its geometric interpretation.

Marginal Cost

In economics, we can of course measure or quantify *cost*, and, in general, cost is expressed as some function of quantity. Given a cost function $C(q)$ we can get the derivative of the cost with respect to quantity produced to obtain the **marginal cost function**, which we denote as $C'(q)$ which gives us the rate of change in cost per unit produced.

For example, suppose that the total cost in pesos incurred each week by a manufacturing firm for q units of a product is given by the total cost function

$$C(q) = 8000 + 200q - 0.2q^2$$

The marginal cost can then be obtained by taking the derivative with respect to q that will yield

$$C'(q) = \frac{dC(q)}{dq} = 200 - 0.4q$$

Therefore, each additional q produced will yield an increase in the total cost equal to $200 - 0.4q$, ceteris paribus.

1.2.6 Product Rule

If f and g are both differentiable functions, and if h is a function defined by $h(x) = f(x)g(x)$, then the derivative of $h(x)$ which is $h'(x)$ can be obtained using the form below

$$h'(x) = f'(x)g(x) + g'(x)f(x) \tag{4}$$

For example, say you are given with the function $h(x) = (x + 5)(x^2 + 1)$. From here, we can assign $f(x) = (x + 5)$ and $g(x) = (x^2 + 1)$.To derive this function, simply do

$$h'(x) = \underbrace{(1)}_{f'(x)}\underbrace{(2x+1)}_{g(x)} + \underbrace{2x}_{g'(x)}\underbrace{(x+5)}_{f(x)}$$

$$h'(x) = (2x + 1) + 2x(x + 5) = 2x + 1 + 2x^2 + 10x = 2x^2 + 10x + 1$$

Marginal Revenue Function

The revenue function is defined as $R(q_j) = p(Q)q_j$ for a perfectly competitive market. If the firm is a monopolist, we have $Q = q_j = q$. We define the **marginal revenue** as the revenue earned by a firm for producing an additional unity of a good. The *marginal revenue function* is the rate of change in revenue per unit of the good sold, and is obtained by differentiating the revenue function with respect to q. To obtain this, we can merely use the product rule (for cases of perfect competition)

$$MR_{PC} = R'(q) = p'(q)q + p(q)$$

1.2.7 Quotient Rule

If f and g are both differentiable functions, and if h is a function defined by $h(x) = \frac{f(x)}{g(x)}$, then it holds that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \tag{5}$$

For example, say we were given with the function $h(x) = \frac{x}{2x-4}$. First, lets treat $f(x) = x$ and $g(x) = 2x - 4$. Following the quotient rule

$$h'(x) = \frac{(2x-4)(1) - (x)(2)}{(2x-4)^2} = \frac{2x - 4 - 2x}{4x^2 - 16x + 16} = \frac{-4}{4x^2 - 16 + 16}$$

Get the derivative of $f(x) = \frac{x^2}{(x^2+3)}$ with respect to x

1.2.8 Chain Rule

If f and g are both differentiable functions, and if h is a function defined by $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x))g'(x) \tag{6}$$

For example. say you were asked to get the derivative of $h(x) = (3x + 1)^2$ with respect to x using the Chain Rule. Say that we think of $3x + 1$ as $g(x)$. To solve using the chain rule, this translates into

$$h'(x) = 2 \cdot (3x + 1)^{2-1} \cdot \frac{dg(x)}{dx}(3x + 1)$$
$$h'(x) = 2(3x + 1) \cdot (3) = 6(3x + 1) = 18x + 6$$

Alternatively, we can factor this binomial out and take the derivative term by term instead of using the chain rule since the power is not to high (not so complicated to get). When we factor, we get that $h(x) = 9x^2 + 6x + 1$. Taking the derivative of this function yield

$$\frac{dh(x)}{dx} = h'(x) = 18x + 6$$

Find the derivative of the following functions with respect to x using the chain rule.

1. $f(x) = (x^3 + 2)^5$
2. $f(x) = \left(\frac{1}{x} + 1\right)^2$

1.2.9 Derivatives of Exponential and Logarithmic Functions

If $f(x) = e^x$, then the derivative of $f(x)$ is merely the original function.

$$f'(x) = e^x \tag{7}$$

For example, say we were given with the function $f(x) = e^{2x-3}$. To solve this, we need to apply the rule above and also chain rule.

$$f'(x) = e^{2x-3} \cdot \frac{d}{dx}(2x - 3) = e^{2x-3} \cdot 2 = 2e^{2x-3}$$

If $f(x) = \ln x$, then the derivative of the function is merely $\frac{1}{x}$

$$f'(x) = \frac{1}{x} \tag{8}$$

Consider the function $f(x) = \ln(x^2 + 3x + 7)$, deriving this needs the rule above and also the chain rule.

$$f'(x) = \frac{1}{x^2 + 3x + 7} \cdot \frac{d}{dx}(x^2 + 3x + 7) = \frac{2x + 3}{x^2 + 3x + 7}$$

If a is a constant number and $f(x) = a^x$, then

$$f'(x) = a^x \ln a \tag{9}$$

Say we were given with $y = 7^x$, we can derive this function with respect to x and yields

$$y' = 7^x \ln(7)$$

Find the derivative of the following functions

1. $y = e^{-x}$
2. $y = 3e^{1-x}$
3. $y = xe^{1-x}$
4. $y = e^{x^2}$
5. $y = \ln(e^x + e^{-x})$
6. $y = \frac{1}{2^x+e^x}$

1.2.10 Implicit Differentiation

When variables y and x are related by an equation of the form $F(x, y) = 0$, we say that y is an **implicit function** of x. With implicit functions, it may be difficult to express y as an explicit function of x, or $y = f(x)$. Thus, **implicit differentiation** may be useful.

For instance, say we had the function $y^3 + y^2 + x^2 = 2$

Differentiating term by term with respect to x, we get

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} + 2x = 0$$

Rearranging

$$\frac{dy}{dx}(3y^2 + 2y) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{3y^2 + 2y}$$

Find $\frac{dy}{dx}$ by implicit differentiation

1. $x^2 + y^2 - 3 = 0$
2. $x^2 + y^2 = xy$
3. $y + x^2y - 1 = 0$

Growth Rate of a Variable

We define the **growth rate of a variable** $x(t)$ that varies across one time period as

$$g_x = \frac{x(t+1) - x(t)}{x(t)} \tag{10}$$

In general, we compute the growth rate of $x(t)$ across several time periods as

$$g_x = \frac{x(t+\Delta t) - x(t)}{\Delta t} \cdot \frac{1}{x(t)}$$

Letting h approach zero, or computing the growth rate of $x(t)$ over a very small time period, we redefine the growth rate of a variable as

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \cdot \frac{1}{x(t)} = \frac{x'(t)}{x(t)} \approx g_x$$

Elasticity

From basic microeconomics, the **elasticity of demand with respect to price** is defined as

$$\varepsilon = -\frac{\frac{q_2 - q_1}{q_1}}{\frac{p_2 - p_1}{p_1}} = -\frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}}$$

In the form above, Δq is the change in quantity demanded and Δp is the change in prices. We say that the good is **price inelastic** if $|\varepsilon| < 1$, **price elastic** if $|\varepsilon| > 1$, and **unitary inelastic** if $|\varepsilon| = 1/$

Rewriting the formula for elasticity yields

$$\varepsilon = -\frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}} = -\frac{\Delta q}{\Delta p} \cdot \frac{p}{q}$$

Letting the change in prices approach 0, we obtain

$$-\frac{p}{q} \lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} = \frac{p}{q} q' \approx \varepsilon$$

However, most microeconomic textbooks will use the format

$$\varepsilon = -\frac{p}{q} \cdot \frac{dq}{dp} \tag{11}$$

For example, suppose that the demand for a commodity is given by $q = 120 - 4p$ where q is the quantity demanded and p is the price. We first determine the price elasticity of demand which is the point elasticity of q with respect to p

$$\varepsilon = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p}{q} \cdot -4$$

If you were asked to determine and interpret the price elasticity when the price is say $p = 10$, you would need to plug this in to the demand function first to get q.

$$q = 120 - 4(10) = 120 - 40 = 80$$

Plugging this in to our elasticity form

$$\varepsilon = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{10}{80} \cdot -4 = \frac{40}{80} = \frac{1}{2}$$

Since $\varepsilon = 1/2 < 1$, we can sat that the demand for this commodity is price inelastic.

> Using the same demand function used, determine and interpret the price elasticity of demand at $p = 20$ and at $p = 30$. What do you observe?

1.3 Higher Order Derivatives

1.3.1 Second Order Derivative

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function is called the **second order derivative of** f because it is the derivative of the derivative of f. We say that f is **twice differentiable**.

Consider the function $y = x^3 - 3x^2 + 1$. We take the first order derivative of this function as

$$\frac{dy}{dx} = y' = 3x^2 - 6x$$

To get the second order derivative, we take the derivative of this (first-order) derivative as what you can see below.

$$\frac{d^2y}{dx^2} = y'' = 6x - 6$$

Get the first and second order derivative of the following functions

1. $y = 7x^2 + 2x^3 + 78$
2. $y = (2x^2 + 2)^2$
3. $y = 2x - 1$

1.3.2 Continously Differentiable Function

In more general notation, we have $f^{(n)}(x)$ called the n^{th} **order derivative** of f with respect to x. If f is a differentiable function and its derivative is a continuous function, we say that f is a **continuously differentiable function**.

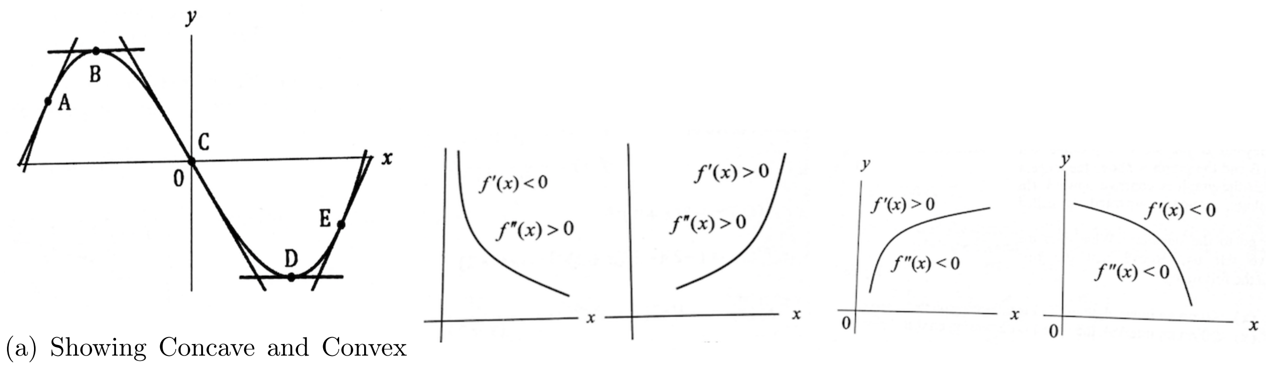
If f s twice differentiable and its second order derivative is also a continuous function, we say that f is a **twice continuously differentiable function**.

Derive the following functions as far as you can until you get a derivative equal to zero

1. $y = 7x^2 + 2x^3 + 78$
2. $y = 9x^8 + 8x^7 + 7x^6 + 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$

1.3.3 Implication of the Second Derivative

Consider the graph of a function f. From point A to B, the slope is positive but decreasing until it becomes zero at B. Beyond B, the slope is negative and decreasing up to C. We then say that the graph is **concave** from A to C. From C to D, the slope is negative but increasing until it becomes zero at D. Beyond D, the slope is positive and increasing up to E. We then say that the graph is **convex** from C to E.



(a) Showing Concave and Convex in f

(b) Convex Graphs

(c) Concave Graphs

Figure 3: Shape of Functions

For example, consider the function $f(x) = x^2 + 2x + 1$. We can obtain the first and second order of this function as

$$f'(x) = 2x + 2$$
$$f''(x) = 2 > 0$$

To identify on what range the function is concave or convex, we can plug in various values of x, When $x > -1$, then $f'(x) > 0$ which means that the graph on these points is convex coming from a higher slope going to a lower slope. When $x = -1$, $f(x) = 0$ which means that the graph is at a stationary point in this range. When $x < -1$, then $f'(x) < 0$ which means that the graph at this point is convex still.

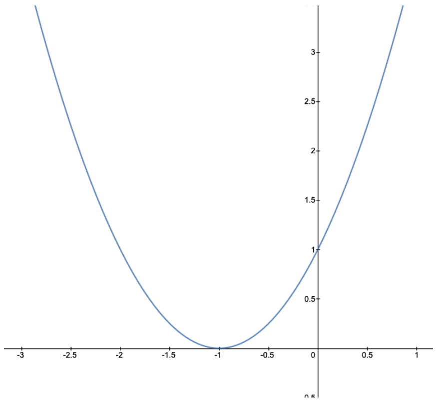


Figure 4: Graph of $f(x) = x^2 + 2x + 1$

1.4 Differential

If $y = f(x)$, where f is a differentiable function, then the differential dy is defined by the equation

$$dy = f'(x)dx$$

In the form above, $f'(x)$ is the derivative of y with respect to x and $dx = \Delta x$ is the change in x. The process of obtaining the differential of y is called **total differentiation**. The differential measures the estimated change in y given a change in x, that is $\Delta y \approx dy$.

1.4.1 Economic Application using National Income Accounting

Given a three-sector economy $Y = C + I + G$ where household consumption is determined by a linear consumption function that depends on income, $C(Y) = \alpha + \beta Y$. Further assume that consumption and investment are exogenous to the model and only government expenditure is endogenous. The equilibrium output of this basic economy is just merely

$$Y^* = Y^*(G) = \frac{\alpha + I + G}{1 - \beta}$$

If government expenditure changes, we can compute the multiplier effect by totally differentiating Y^*.

$$dY^* = Y^*(G)dG = \frac{1}{1 - \beta}dG$$

In the expression above, we refer to $\frac{1}{1-\beta}$ as the multiplier effect. To illustrate, suppose that the consumption function $C(Y) = 5 + 0.5Y$ and investments are fixed at 50. Then the multiplier is equal to 2. If If initial government expenditures was 100 and an expansionary fiscal policy increases it to 101, then the change in government expenditures is $dG = 1$. This means that the change in equilibrium output is $dY^* = 2$. Note that the change dY is just the multiplier (which is 2) multiplied to the change in G which was dG. We can test this out by actually computing for this change.

$$Y^*(101) - Y^*(100) = \frac{5 + 50 + 101}{0.5} - \frac{5 + 50 + 100}{0.5} = \frac{156}{0.5} - \frac{155}{0.5} = 312 - 310 = 2$$

1.4.2 Property of a Differential

If $y = f(x)$, where f is a differentiable function, and the change in x is very small, then the differential of y is close to the actual change in y. That is:

$$\lim_{dx \to 0} \Delta y = dy \tag{12}$$

Graphically, we illustrate the differential as

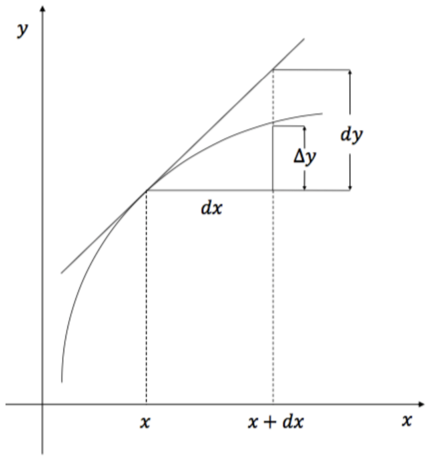


Figure 5: Illustrating the Differential

In our linear example in the three sector economy, we determined that $dy = \Delta Y$. However, when functions are non-linear, this may not always be the case. We may only get an approximation as in $dy \approx \Delta Y$

Theorem 1.3. *As the differential dx gets smaller, the difference between the actual change Δy and the differential dy goes closer and closer to zero.*

Proof. Let $y = f(x) = x^2$ and let $x = 2$

If $dx = 0.1 = \Delta x$, then

$$\Delta y = f(2.1) - f(2) = (2.1)^2 - (2)^2 = 0.41$$
$$dy = 2xdx = 2(2)(0.1) = 0.4$$
$$\therefore \Delta y - dy = 0.41 - 0.40 = 0.01$$

If $dx = 0.01 = \Delta x$, then

$$\Delta y = f(2.01) - f(2) = (2.01)^2 - (2)^2 = 0.0401$$
$$dy = 2xdx = 2(2)(0.01) = 0.04$$
$$\therefore \Delta y - dy = 0.0001$$

Thus, as the differential dx becomes smaller, the differential dy gets closer to the actual change Δy □

1.5 Differentiation with Several Variables

1.5.1 Partial Derivatives

It is myopic to believe that all functions are just functions with one variable. Let $z = f(x, y)$. The **partial derivative of f with respect to x** at (x, y) is a function f_x defined by the form below if the limit exists.

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}|_{y=\bar{y}}$$

Similarly, the **partial derivative of f with respect to y** at (x, y) is a function f_y is defined by the form below if the limit exists

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta y}|_{x=\bar{x}}$$

Likewise, we also use the following notations for the partial derivative

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y)$$
$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y)$$

To find the partial derivative with respect to x, that is, $f_x(x, y)$, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find $f_y(x, y)$ or the partial derivative with respect to y, regard x as a constant and differentiate.

For example, say we have $f(x, y) = 2x^2 + y^3 + xy$. Differentiate the function with respect to both x and y. Doing this should be straightforward

$$f_x(x,y) = \frac{\partial f}{\partial x} = 4x + y$$
$$f_y(x,y) = \frac{\partial f}{\partial y} = 3y^2 + x$$

Obtain the partial derivative with respect to each variable

1. $f(x, y) = 3y^2 e^x + y^2$
2. $f(x, y, z) = xyz^2 + 3xyz + z^3$
3. $f(\alpha, \beta) = \alpha^{\varepsilon}\beta^{1-\varepsilon}$

1.5.2 Higher-Order Partial Derivatives

Since the partial derivative is a function, we may also define its partial derivatives. These are called **second-order partial derivatives** and are denoted as follows

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x} \cdot \left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y} \cdot \left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x} \cdot \left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \cdot \left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

We refer to f_{xx} and f_{yy} as the **direct second order partial derivatives**. We refer to f_{xy} and f_{yx} as the **cross partial derivatives**.

Theorem 1.4. *The cross partial derivatives are necessarily equal to each other (Young's Theorem)*

Instead of showing a formal proof, let us use an example. Consider the function $z = f(x, y) = x^3 + x^2y^2 + y^3$. We obtain the first-order partial derivatives

$$f_x(x, y) = 3x^2 + 2xy^2$$

$$f_y(x, y) = 2x^2y + 3y^2$$

We obtain the cross partial derivatives

$$f_{xy} = 4xy$$

$$f_{yx} = 4xy$$

Hence, Young's Theorem holds since the cross partial derivatives are necessarily equal.

Find the second order direct and cross partial derivatives of the following functions. Demonstrate that Young's Theorem holds.

1. $z = f(x, y) = xe^y$
2. $z = f(x, y) = 3xy + x^5 + y^3$

1.5.3 Marginal Functions in Economics

Consider a production function with capital and labor as its factors of production, $F(K, L)$. Then the partial derivative $F_K(K, L)$ is called the **marginal productivity of capital,** which measures the rate of change of production with respect to the amount of capital used with the level of labor held fixed, and $F_L(K, L)$ is called the **marginal productivity of labor**, which measures the rate of change of production with respect to the amount of labor used with the level of capital held fixed.

For example, consider a Cobb-Douglas Production function $F(K, L) = AK^\alpha L^\beta$. We obtain the marginal product of capital and the marginal product of labor by just getting the first order partial derivative with respect to each input.

$$\frac{\partial F(K, L)}{\partial K} = \alpha AK^{\alpha-1}L^\beta = f_K(K, L) = MP_K$$
$$\frac{\partial F(K, L)}{\partial L} = \beta AK^{\alpha}L^{\beta-1} = f_L(K, L) = MP_L$$

Notice that under a reasonable domain where $K > 0$ and $L > 0$, then both MP_K and MP_L are greater than zero or strictly positive. This is because for the most part, an increase in labor (or capital), holding the other input fixed (ala ceteris paribus), will increase the output $F(K, L)$.

Despite this being positive, there still exists diminishing returns. The **law of diminishing returns** states that after a certain point, when additional units of a variable input are added to fixed inputs, the *marginal product of the variable input declines.* Given a production function $F(K, L)$ with marginal productivity of capital and labor, f_K and f_L, the law of diminishing marginal productivity is exhibited by mathematically as f_{KK} and f_{LL} are less than zero.

Theorem 1.5. *The Law of Diminishing Marginal Productivity requires that for a typical production function,* $f_{xx}, f_{jj} < 0 \ \ \forall x, j > 0$

From our example, we know that $MP_K = \alpha A K^{\alpha-1} L^{\beta} = f_K(K,L)$ and $MP_L = \beta A K^{\alpha} L^{\beta-1} = f_L(K,L)$. Further note that $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Obtaining the second order direct partial derivatives

$$f_{KK} = (MP_K)_K = \alpha(\alpha-1)K^{\alpha-2}L^{\beta} = (\alpha^2-\alpha)K^{\alpha-2}L^{\beta} < 0 \ \ \forall K, L > 0$$
$$f_{LL} = (MP_L)_L = \beta(\alpha-1)K^{\alpha}L^{\beta-2} = (\beta^2-\beta)K^{\alpha}L^{\beta-2} < 0 \ \ \forall K, L > 0$$

Similarly, we can apply these constructs in the theory of consumer behavior. Consider the utility function $U(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$. In this utility function, x_1 corresponds to the amount of good 1 consumed and x_2 corresponds to the amount of good 2 consumed. In general, utility or satisfaction is defined by the consumption of goods. Let $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$

Theorem 1.6. *By an axiom of consumer behavior, a rational consumer is always non-satiated. Hence, each additional unit consumed, none less the other, will always increase the total utility. That is,* $MU_i > 0 \ \forall i$.

If we take the first-order derivative of the utility with respect to each good

$$MU_1 = \frac{\partial U(x_1, x_2)}{x_1} = f_{x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} > 0 \ \ \forall x_1, x_2 > 0$$

$$MU_2 = \frac{\partial U(x_1, x_2)}{x_2} = f_{x_2} = \beta x_1^{\alpha} x_2^{\beta-1} > 0 \ \ \forall x_1, x_2 > 0$$

As with production, there is also diminishing returns in utility. The **law of diminishing marginal utility** states that the more of any one good is consumed in a given period, the less satisfaction (utility) is generated by consuming each additional (marginal) unit of the same good. If we have a utility function $U(x_1, x_2)$, then the assumption of **diminishing marginal utility of good** i is stated mathematically as $U_{x_i x_i} < 0$

Theorem 1.7. *The second order cross partial derivatives of the utility function should be negative by the law of diminishing marginal utility. That is,* $f_{x_i x_i} < 0 \ \ \forall x_i > 0$

$$f_{x_1 x_1} = (MU_1)_{x_1} = \frac{\partial^2 U(x_1, x_2)}{x_1^2} = \alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta} < 0 \ \ \forall x_1, x_2 > 0$$

$$f_{x_2 x_2} = (MU_2)_{x_2} = \frac{\partial^2 U(x_1, x_2)}{x_2^2} = \beta(\beta-1)x_1^{\alpha}x_2^{\beta-2} < 0 \ \ \forall x_1, x_2 > 0$$

Using the production and utility functions, prove the properties and theorems described thusfar by taking the first and second order partial derivative.

1. $U(x_1, x_2) = 2x_1x_2 + 3x_1^2 + 3x_2^2$
2. $F(K, L) = K^{0.5}L^{0.5}$

1.5.4 Total Differentials

Let $z = f(x, y)$. The partial derivative f_x determines the effect of x on z while holding y constant. What if both x and y are changing? The joint effect would then be obtained by taking the sum of their individual effects. Thus, we extend the concept of the differential to functions of two variables. Given the differentials dx and dy, the total differential dz is given by

$$dz = f_x dx + f_y dy$$

In the form above, dz approximates change in z and is due to the sum of the changes dx and dy.

For example, say we want to take the total differential of the function $z = y^2 - 4xy$. To obtain this would be straightforward

$$dz = f_x dx + f_y dy = -4y\ dx + (2y - 4x)\ dy$$

Take the total differential of the following functions

1. $z = ln(xy)$
2. $z = 4x^2 + 8xy^2 + 9x^2y$

2 Applications of Differentiation

2.1 Taylor Approximation

Polynomial approximations of differentiable functions are useful in mathematical analysis. Taylor's Theorem states that a differentiable function can be approximated by a polynomial. If f is an n-times differentiable function, then the Taylor approximation to f about $x = x_0$ is:

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{13}$$

We refer to the linear approximation as the first-order Taylor approximation and the quadratic approximation as the second- order Taylor approximation.

2.1.1 Linear Approximation

If f is a differentiable function, then the linear approximation (first-order Taylor approximation) to f about $x = x_0$ is:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Note that when $x = x_0$, the function $f(x)$ and the tangent line $f(x_0) + f'(x_0)(x - x_0)$ have the same slope.

2.1.2 Quadratic Approximation

If f is a twice-differentiable function, then the quadratic approximation (second-order Taylor approximation) to f about $x = x_0$ is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

For example, say you were given with the function $f(x) = e^x$. Find the first and second order Taylor polynomial about $x_0 = 0$.

The first-order Taylor polynomial about $x_0 = 0$ is

$$P_1(x) = e^0 + e^0(x - 0)$$
$$P_1(x) = 1 + x$$

The first-order Taylor polynomial about $x_0 = 0$ is

$$P_2(x) = e^0 + e^0(x - 0) + \frac{1}{2}e^0(x - 0)^2$$
$$P_2(x) = 1 + x + \frac{1}{2}x^2$$

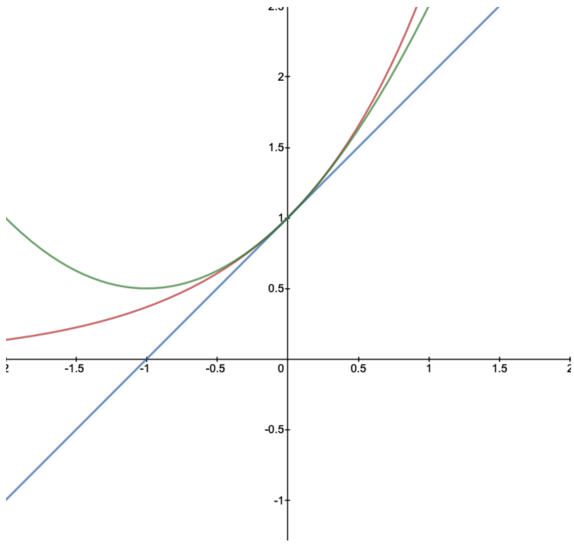


Figure 6: Illustrating the First and Second Order Taylor Polynomial

In the figure above, we see the **red** line corresponds to the original function $f(x)$ while the **blue** line corresponds to the linear Taylor approximation $P_1(x)$. Lastly, the **green** corresponds to the quadratic Taylor approximation $P_2(x)$.

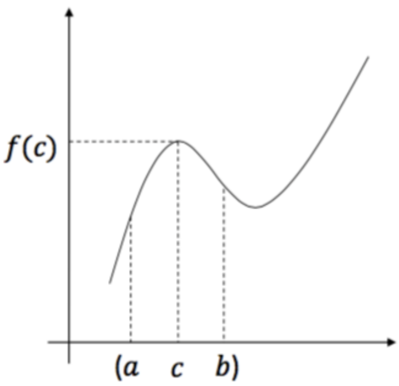
Find the first-and second order Taylor polynomial for the following functions

1. $f(x) = \ln(1 + x)$ about $x_0 = 0$
2. $f(x) = x^3 - 10x^2 + 6$ about $x_0 = 3$

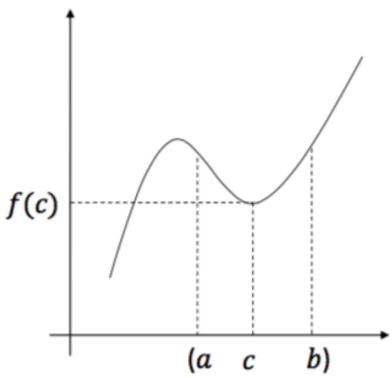
2.2 Optimization

2.2.1 Local Maxima and Minima

The function f has a **local (relative) maximum value** at the number c if there exists an open interval containing c ,on which f is defined, such that $f(c) \geq f(x)$ for all x in this interval. Conversely, the function f has a **local (relative) minimum value** at the number c if there exists an open interval containing c,on which$f$ is defined, such that $f(c) \leq f(x)$ for all x in this interval.



(a) Local Maximum

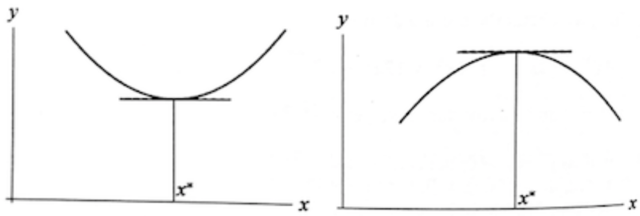


(b) Local Minimum

Figure 7: Illustrating a Local Maxima or Minima

Theorem 2.1. *If f has a local minimum or a local maximum at x^*, and if $f'(x^*)$ exists, then $f'(x^*) = 0$. (Fermat's Theorem)*

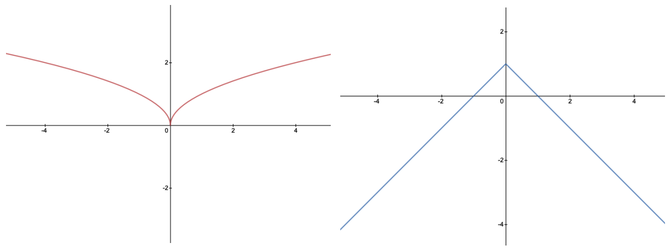
Notice that at point c, whether a local minimum or maximum, the slope of the tangent line (derivative) at these points are zero. More formally, the tangents at $(x^*, f(x^*))$ and at $(x^*, g(x^*))$ below are horizontal; hence, $f'(x^*)$ and $g'(x^*)$ are equal to 0.



(a) Local Minimum (b) Local Maximum

Figure 8: Illustrating Fermat's Theorem

Note that the Fermat's Theorem requires differentiability at the point x^*. When the derivative at x^* does not exist, it is still possible for a function to have a local maximum or a local minimum at x^*, but the derivative will not be equal to zero at x^*.



(a) Graph of $y = \sqrt{|x|}$ (b) Graph of $y = 1 - |x|$

Figure 9: Special Cases where $f'(x) \neq 0$

If x^* is a number in the domain of function f, and if $f'(x*) = 0$, then x^* is a **critical number** of the function and we call that point $(x^*, f(x^*))$ a **stationary point**.We can rephrase Fermat's theorem as follows: If f has a local maximum or minimum at x^*, then x^* is a critical number of f.

2.2.2 The First-Derivative Test

If the derivative of a function $f'(x)$ changes sign around a critical point x^*, the function is said to have a local (relative) optimum at that point.

- If $f'(x)$ changes from positive to negative, the function has a *local maximum* at the critical point x^*.
- If $f'(x)$ changes from negative to positive, the function has a *local minimum* at the critical point x^*.
- If $f'(x)$ does not change in sign, then the function has neither a local maximum nor local minimum at the critical point x^*.

This technique that is used to determine local maximum or minimum values is called the **first-derivative test** for local optima. Take note that $f'(x^*) = 0$ is a **necessary (first-order) condition** for $f(x^*)$ to be an optimal value of the function.

For example., determine the local optima of $f(x) = \frac{1}{3}x^3 - 4x$ by using the First Derivative Test. To start, we find the first derivative of the function first.

$$f'(x) = x^2 - 4 = (x + 2)(x - 2)$$

Setting $f'(x) = 0$, we can obtain the critical values

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Therefore, the critical numbers are $x^* = 2$ or $x^* = -2$. Since we have two critical points, we can generally observe three potential ranges. These are $x < -2$, $-2 < x < 2$, and $x > 2$. Determining at which critical value is a maximum/minimum lie in identifying the behavior at each point and analyzing.

If $x < -2$, say let's use $x = -3$

$$f'(x) = x^2 - 4 = (-3)^2 - 4 = 9 - 4 = 5 > 0$$

Since the sign is positive, then it means that the points immediately before the point of -2 have a positive slope.

If $-2 < x < 2$, say let's use $x = 0$

$$f'(x) = x^2 - 4 = (0)^2 - 4 = -4 < 0$$

Since the sign is negative, then it means that within the range, the slope of the function is negative, which suggests that the point it is approaching to (which is $x = 2$) is a minimum point. To confirm this assertion, let us investigate when $x > 2$.

If $x > 2$, say let's use $x = 3$

$$f'(x) = x^2 - 4 = (3)^2 - 4 = 5 > 0$$

Hence, for this range, the function is positively sloped. The signs show us that f has a local maximum at $x^* = -2$ and a local minimum at $x^* = 2$. We graph this function for confirmation.

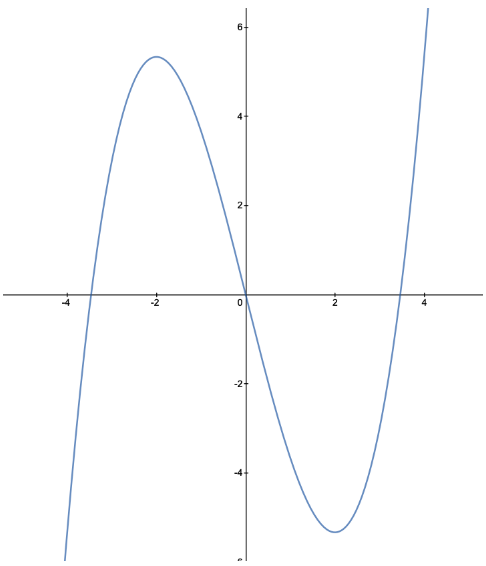


Figure 10: Illustrating $f(x) = \frac{1}{3}x^3 - 4x$

Determine the local optima of the following functions using the First Derivative Test

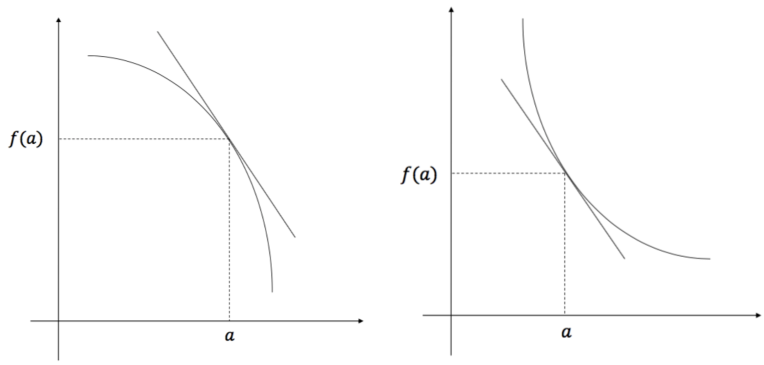
1. $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3$
2. $f(x) = x^3 - 3x - 4$

2.3 Concavity and Convexity

2.3.1 Strict Concavity and Convexity

The graph of a function f is said to be *strictly concave* at the point $(a, f(a))$ if $f'(a)$ exists and if there is an open interval I containing a such that for all values of $x \neq a$ in I, the point $(x, f(x))$ on the graph is **below** the tangent line to the graph at $(a, f(a))$. In other words, a function f is said to be strictly concave on the interval $[a, b]$ if and only if the line segment joining any two points of the graph of f lies **entirely below** the graph. Recall that concavity (or convexity) in an interval is determined by the sign of the second derivative. Whenever $f''(x) < 0$, the graph of f for that interval is **strictly concave** in shape.

Conversely, the graph of a function f is said to be *strictly convex* at the point $(a, f(a))$ if $f'(a)$ exists and if there is an open interval I containing a such that for all values of $x \neq a$ in I ,the point $(x, f(x))$ on the graph is **above** the tangent line to the graph at $(a, f(a))$. In other words, a function f is said to be strictly convex on the interval $[a, b]$ if and only if the line segment joining any two points of the graph of f lies entirely **above** the graph. Recall that concavity (or convexity) in an interval is determined by the sign of the second derivative. Whenever $f''(x) > 0$, the graph of f for that interval is **strictly convex** in shape.



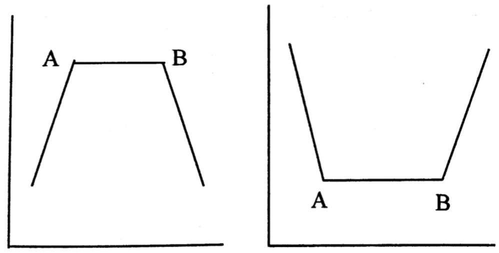
(a) Strict Concavity (b) Strict Convexity

Figure 11: Illustrating Strict Concavity and Convexity

2.3.2 Weak Concavity and Convexity

The graph of a function f is said to be **concave or weakly concave** at the point $(a, f(a))$ if $f'(a)$ exists and if there is an open interval I containing a such that for all values of $x \neq a$ in I, the point $(x, f(x))$ on the graph is **on or below** the tangent line to the graph at (a, f (a)). In other words, a function f is said to be concave or weakly concave on the interval $[a, b]$ if and only if the line segment joining any two points of the graph of f lies on or below the graph. Recall that concavity (or convexity) in an interval is determined by the sign of the second derivative. Whenever $''(x) \leq 0$, the graph of f for that interval is convex or weakly concave in shape.

On the other hand, the graph of a function f is said to be **convex or weakly convex** at the point $(a, f(a))$ if $f'(a)$ exists and if there is an open interval I containing a such that for all values of $x \neq a$ in I, the point $(x, f(x))$ on the graph is **on or above** the tangent line to the graph at (a, f (a)). In other words, a function f is said to be convex or weakly convex on the interval $[a, b]$ if and only if the line segment joining any two points of the graph of f lies on or above the graph. Recall that concavity (or convexity) in an interval is determined by the sign of the second derivative. Whenever $''(x) \geq 0$, the graph of f for that interval is convex or weakly convex in shape.



(a) Weak Concavity (b) Weak Convexity

Figure 12: Illustrating Weak Concavity and Convexity

2.3.3 Slope and Shape Using the First and Second order Derivatives

The first order derivative may generally be considered to imply the **slope** of $f(x)$. If $f'(x) > 0$, this means that $f(x)$ increases as x increases. Hence, in this case, the function has a positive slope. Conversely, if $f'(x) < 0$, this means that $f(x)$ decreases as x increases. This implies that the function has a negative slope. If $f'(x) = 0$, then the function is at an extremum or an inflection point.

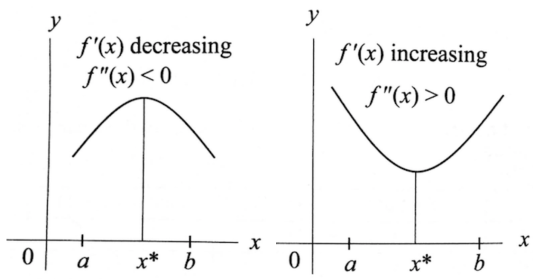
The second-order derivative implies the **shape** of $f(x)$. If $f''(x) > 0$, then the slope $f'(x)$ is increasing as x increases. Conversely, if $f''(x) < 0$, then the slope $f'(x)$ is decreasing as x increases. Lastly, if $f''(x) = 0$, then the slope $f'(x)$ is constant.

2.4 The Second Derivative Test

If f is twice differentiable at x^*, we can extend the implications of the first order derivative test to determine whether the function is at a local minimum or a maximum without having to do the rigorous interval testing.

- If $f'(x^*) = 0$ and $f''(x^*) > 0$, then f has a local **minimum** at x^*
- If $f'(x^*) = 0$ and $f''(x^*) < 0$, then f has a local **maximum** at x^*
- If $f'(x^*) = 0$ and $f''(x^*) = 0$, then f, then the test is inconclusive

The second-order derivative test provides us a **sufficient (second-order) condition** for $f(x^*)$ to be an optimal value.



(a) Showing a Maximum (b) Showing a Minimum

Figure 13: Illustrating the Second Derivative Test

For example, say you were tasked to determine the local optima of $f(x) = x + \frac{1}{x}$. First, we should get the first order derivative and equate to zero per the procedure of the first-derivative test.

$$f'(x) = 1 - \frac{1}{x^2} = 0$$
$$x = \pm 1$$

After which, we get the second order derivative

$$f''(x) = \frac{2}{x^3}$$

We evaluate the second derivative at thee critical values. At $x^* = -1$, $f''(-1) = \frac{2}{(-1)^3} = -2 < 0$; hence, f has a local maximum at $x^* = -1$. Meanwhile, at $x^* = 1$, then $f''(1) = \frac{2}{(1)^3} = 2 > 0$; hence, f has a local minimum at $x^* = 1$.

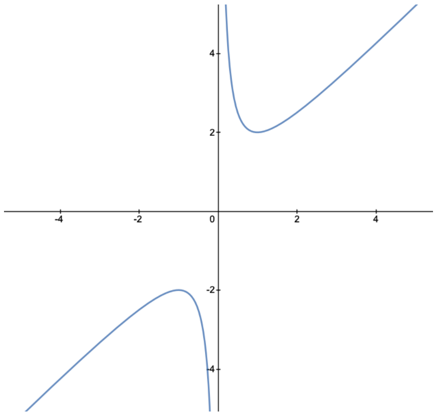


Figure 14: Illustrating $f(x) = x + \frac{1}{x}$

Determine the local optima of the following functions using the First Derivative Test and Second Derivative Test

1. $f(x) = \frac{1}{3}x^3 - 4x$
2. $f(x) = x^3 - 3x - 4$

2.4.1 Inflection Point

In some cases, the Second-Derivative Test will not work when identifying whether a critical point is a local maximum or a minimum, i.e. it will yield an inconclusive result ($f''(x^*) = 0$) even though the critical point is actually a maximum or a minimum. In this case, you will have to use the First Derivative Test to check for maximum or minimum using the regular interval testing.

The point $(a, f(a))$ is a **point of inflection** of the graph of f if the graph has a tangent line at that point where either:

- $f''(x) < 0$ if $x < a$ and $f''(x) > 0$ if $x > a$; or
- $f''(x) < 0$ if $x > a$ and $f''(x) > 0$ if $x < a$

In other words, a point is an inflection point if f is continuous there and the curve changes from concave to convex or from convex to concave at the point. At the point of inflection, if $f''(a)$ exists, then $f''(a) = 0$ or $f''(a)$ is undefined.

For example, say you were given $f(x) = x^3 - 6x^2 + 9x + 1$. Of course, to get the second order derivative, we would need to get the first order derivative.

$$f'(x) = 3x^2 - 12x + 9$$

Getting the second derivative of this function, we get

$$f''(x) = 6x - 12$$

To get the inflection point, we set this second order derivative equal to zero

$$0 = 6x - 12$$

$$x = 2$$

Therefore, when $x = 2$, the $f''(x) = 0$, suggesting that the function is at an inflection point. If $x < 2$, then $f''(x) < 0$, which means that for this range, the graph is concave. Conversely, when $x > 2$, then $f''(x) > 0$, which means that the graph is convex in this domain. To find the specific inflection point, we need the coordinate value of $f(x)$. This can be done by evaluating $f(x)$ at the inflection x value computed which is $x = 2$.

$$f(2) = (2)^3 - 6(2)^2 + 9(2) + 1 = 8 - 24 + 18 + 1 = 3$$

Hence, the inflection point for this function is at point $(2, 3)$. We verify this by graphing out the function.

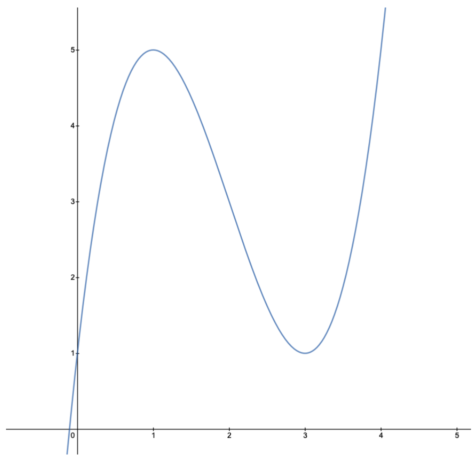
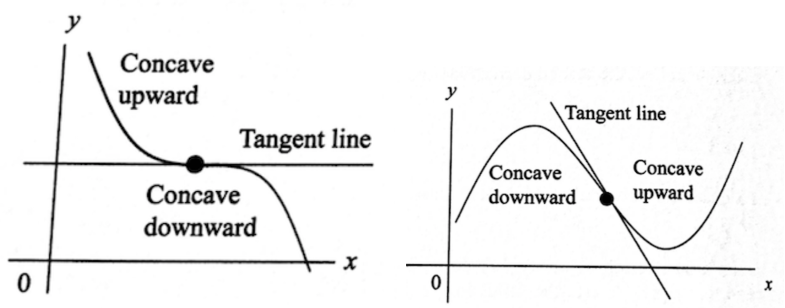


Figure 15: Illustrating $f(x) = x^3 - 6x^2 + 9x + 1$

In general, an inflection point may look like the two illustrations below. We see that our example above corresponds more to the second illustration. The inflection point "essentially" bisects the concave and convex portions of the graph.



(a) Horizontal Tangent Line (b) Diagonal Tangent Line

Figure 16: Illustrating and Inflection Point

Note that $f''(a) = 0$ is a necessary condition for an inflection point. However, it is possible for $f''(a) = 0$ at a point a that is not an inflection point.

Find the inflection points of $f(x) = x^4 - 4x^3 + 10$

2.5 Global Optimum

A function f with domain D has a global (or absolute) minimum at x^* if $f(x^*) \leq f(x)$ for all x in the domain. A function f with a domain D has a global maximum at x^* if $f(x^*) \geq f(x)$ for all x in the domain.

2.5.1 Weierstrauss' Theorem

Theorem 2.2. *If f is continuous on a closed interval $[a, b]$, then f attains a global maximum value $f(c_1)$ and a global minimum value $f(c_2)$ at some numbers c_1 and c_2 within $[a, b]$.*

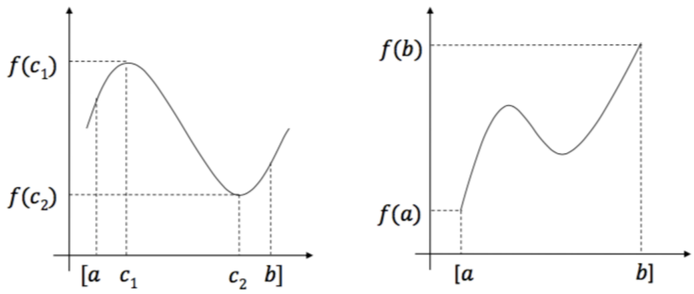


Figure 17: Illustrating Weierstrauss' Theorem

We often refer to the Weirstrauss' Theorem as the *extreme value theorem*. Note that the Extreme Value Theorem fails if the interval is not closed. For example, the function $f(x) = x$ does not have a maximum value on $(0, 1)$. Further note that continuity of the function is required at all points of the interval $[a, b]$. Otherwise, a discontinuity could destroy the conclusion of the theorem. For example, say the function is discontinuous at $x = 0$. As x approaches zero from both sides, $f(x)$ goes up to infinity. In this case, no maximum is attained. In application, when the interval is closed, we not only examine critical values as candidates for optima, but **we also test the endpoints of the interval**.

For example, say you were asked to find the global maxima and minima of $f(x) = x^4 - 2x^2 + 1$ on the closed interval $[2, 2]$. We first use the first derivative test to obtain the critical values.

$$f'(x) = 4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$4x(x - 1)(x + 1) = 0$$

Therefore, the critical values of x are $x^* = -1$, $x^* = 0$, and $x^* = 1$. We now evaluate the function $f(x)$ at these critical values, as well as at the endpoints $x = -2$ and $x = 2$.

Testing the critical values of x

$$f(-1) = (-1)^4 - 2(-1) + 1 = 0$$
$$f(1) = (1)^4 - 2(1) + 1 = 0$$
$$f(0) = (0)^4 - 2(0) + 1 = 1$$

Testing the endpoints

$$f(-2) = (-2)^4 - 2(-2) + 1 = 9$$
$$f(2) = (2)^4 - 2(2) + 1 = 9$$

Thus, in the interval $[-2, 2]$, the global maxima occur at both $x = -2$ and $x = 2$, the global minima occur at $x = -1$ and $x = 1$, and we have a local maximum at $x = 0$. This conclusion is graphed in the figure to follow.

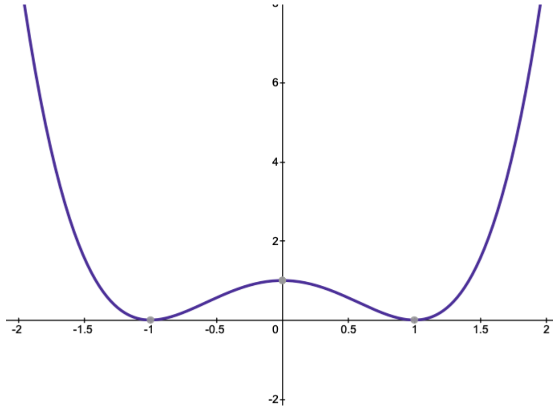


Figure 18: Illustrating $f(x) = x^4 - 2x^2 + 1$ on $[2, 2]$

Find the global maximum and global minimum of $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x - 3$ on $[0, 5]$. Confirm your answers by comparing it with the points on the figure below.

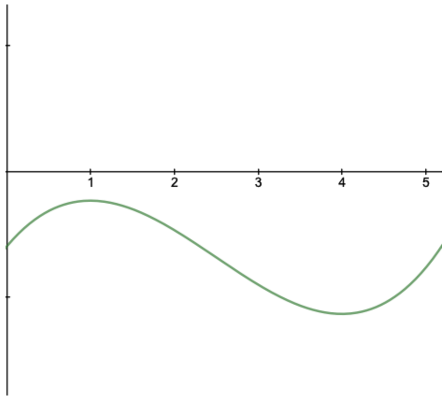
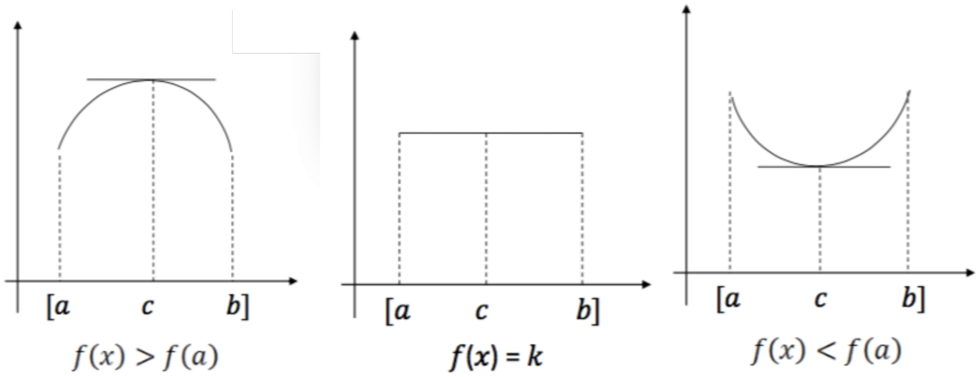


Figure 19: Illustrating $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x - 3$ on $[0, 5]$

2.5.2 Rolle's Theorem

Theorem 2.3. *If f is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and that $f(a) = f(b)$ is true, then there is a number $c \in (a, b)$ such that $f'(c) = 0$.*



(a) For a Concave Case (b) For a Linear Case (c) For a Convex Case

Figure 20: Illustrating Rolle's Theorem

2.6 Profit Maximization

Optimization may be used to solve for the highest profit a firm may be able to attain. Note that this variant is an unconstrained optimization in a univariate case, that is, no constraints and only one variable is of interest.Consider the following example. A firm has a cost function equal to $C(q) = \frac{4}{15}q^3 - 5q^2 + 80q + 80$ where q is the output produced. The inverse demand function for its product is $p = 100 - 2q$, where p is price. Find the maximum profit of the firm

The *profit function* is defined as the difference between revenue and cost. If profit is positive, then the firm is earning positive returns enough to cover costs and have additional funds for reinvestment or dividends.

$$\pi(q) = R(q) - C(q) \tag{14}$$

In our example, we can calculate the profit function as

$$\pi(q) = pq - C(q)$$
$$\pi(q) = (100 - 2q)q - \frac{4}{15}q^3 + 5q^2 - 80q - 80$$
$$\pi(q) = 100q - 2q^2 - \frac{4}{15}q^3 + 5q^2 - 80q - 80$$
$$\pi(q) = -\frac{4}{15}q^3 + 3q^2 + 20q - 80$$

We then calculate $f'(q)$ and set it equal to zero

$$f'(q) = \frac{d\pi}{dq} = -\frac{4}{5}q^2 + 6q + 20$$

Solving for q, we get the critical values $q^* = -5/2$ and $q^* = 10$. We discard the negative solution since there is no negative output. We now check if the critical value $q^* = 10$ gives us a maximum. After this, we solve for the second derivative of $\pi(q)$

$$f''(q) = \frac{d^2\pi}{dq^2} = -\frac{8}{5}q + 6$$

At $q^* = 10$. we have

$$f''(10) = -\frac{8}{5}(1) + 6 = -10 < 0$$

Since $f''(q^*) < 0$, this implies a maximum profit. We can solve for the maximum profit by substituting $q^* = 10$ to our profit function.

$$\pi(q^*) = -\frac{4}{15}(10)^3 + 3(10)^2 + 20(10) - 80 = \frac{460}{3}$$

Therefore, the maximum profit is $\pi^* = \frac{460}{3}$.

Solve the following problems.

- The inverse demand function for a commodity is given by $p = 41 - 0.33q$ and the cost of producing that commodity is $c(q) = -0.02q^2 + 10q + 80$. Find the output level that maximizes profit and verify this indeed yields a maximum. Find the maximum profit as well.

- The total cost of producing x radio sets per day is given by $c(q) = \frac{1}{4}x^2 + 35x + 25$ and the price per set at which they may be sold is $p(x) = 50 - \frac{1}{2}x$. Find the output level that maximizes profit and verify this indeed yields a maximum. Find the maximum profit as well.

3 Integration

3.1 Anti-differentiation and Indefinite Integrals

3.1.1 The Antiderivative

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I. The process of finding all antiderivatives of a function is called **antidifferentiation**.

Lemma 3.1. *If f and g are two functions defined on an interval I, such that $f'(x) = g'(x)$ for all x in I, then there is a constant K such that $f(x) = g(x) + K$ for all x in I*

Theorem 3.2. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.*

To understand the theorem and the lemma, let us start with an example. For example, find the antiderivative of $f(x) = 5$. Since the derivative of $5x$ is 5, then $F(x) = 5x$. Note however, that $G(x) = 5x + 3$ is also an antiderivative of 5. Hence, the general antiderivative of $f(x) = 5$ is $F(x) + C$ or $5x + C$ where C is an arbitary constant.

If F is any antiderivative of f on an interval I, then the most general antiderivative of f on I is called the indefinite integral and denoted as

$$\int_I f(x) = F(x) + C$$

The symbol $\int$ is called the integral sign, $f(x)$ is the *integrand*, x is called the integration variable, and C is the constant of integration. The process of obtaining the indefinite integral is called **integration**

3.1.2 Power Rule

As with differentiation, there are a couple of rules which we follow as well.

$$\int x^n \mathrm{dx} = \frac{1}{n+1} x^{n+1} + C, \text{ where } n \text{ is any real number} \tag{15}$$

For example, say you were tasked to get the integral of x^4.

$$\int x^4 \mathrm{dx} = \frac{1}{5} x^5 + C$$

3.1.3 Constant Number Rule

The integral of a constant c is just cx in addition to some constant of integration

$$\int c \ \mathrm{dx} = cx + C, \text{ where } c \text{ is a constant number} \tag{16}$$

For example, say you were tasked to get the integral of 8.

$$\int 8 \mathrm{dx} = 8x + C$$

3.1.4 Factorization and Separation Rules

When we have a form $cf(x)$ and we want to take the derivative of this expression, we can just opt to multiply c to the integral of $f(x)$.

$$\int cf(x) \ \mathrm{dx} = c \int f(x) \ \mathrm{dx}, \text{ where } c \text{ is a constant number} \tag{17}$$

In terms of separability, if we were tasked to take the integral of a form like $f(x) \pm g(x)$, then this can be done by just getting the integral of $f(x)$ and adding (subtracting) this to the integral of $g(x)$.

$$\int [f(x) \pm g(x)] \text{ dx} = \int f(x) \text{ dx} \pm \int g(x) \text{ dx} \tag{18}$$

For example, say you were asked to get the integral of $4x^2 + 3x$

$$\int 4x^2 + 3x \text{ dx} = 4 \int x^2 \text{ dx} + 3 \int x \text{ dx} = 4 \cdot \frac{1}{3}x^3 + 3 \cdot \frac{1}{2}x^2 + C = \frac{4}{3}x^3 + \frac{3}{2}x^2 + C$$

3.1.5 Logarithmic and Exponential Rules

The integral of e^x is just e^x in addition to the constant of integration. This should be straightforward as the derivative of ^{x} is also e^x.

$$\int e^x \text{ dx} = e^x + C \tag{19}$$

The integral of x^-1 is specifically $ln|x|$ in addition to some constant of integration.

$$\int x^{-1} \text{ dx} = \ln|x| + C \tag{20}$$

The integral of a^x where a is some real constant, the integral is just $\frac{1}{\ln(a)}a^x$ in addition to some constant of integration.

$$\int a^x \text{ dx} = \frac{1}{\ln(a)}a^x + C \tag{21}$$

Obtain the indefinite integral of the following functions

1. $\int x^8$ dx
2. $\int \frac{1}{x^2}$ dx
3. $\int \frac{5}{x}$ dx
4. $\int 7^x$ dx
5. $\int (8e^x + 3x)$ dx
6. $\int (7e^x + 9x^5)$ dx
7. $\int (e^x + x^e)$ dx
8. $\int (4x^3 - 3x^2)$ dx
9. $\int (3x^5 + 4x^{3/2} - 2x^{-1/2})$ dx
10. $\int \left(\frac{4x^{10}+5}{x^7}\right)$ dx

3.2 Area Under a Curve and Definite Integrals

The area problem is finding the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S is bounded by (1) the graph of a continous function, (2) the vertical lines $x = a$ and $x = b$, and (3) the x-axis, The region S is called the area under the graph of f.

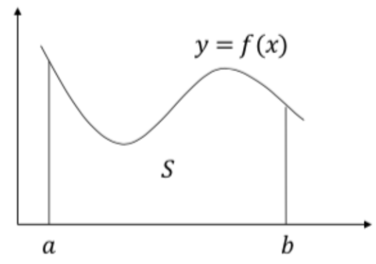


Figure 21: Illustrating the Area Problem

For example, let $f(x) = x^2$ and consider the region S under the graph of f on the closed interval $[0, 1]$. We note that the area of S must be somewhere between 0 and 1 because S is contained in a square of side length equal to 1. To obtain a better approximation of the area of S, we construct four non-overlapping rectangles as follows: We divide the interval $[0, 1]$ into four subintervals of equal length of $\frac{1}{4}$. That is, we form four rectangles, the range of which are $[0, \frac{1}{4}], [\frac{1}{4}, \frac{2}{4}], [\frac{2}{4}, \frac{3}{4}], [\frac{3}{4}, 1]$.

After this, we construct the four rectangles with these subintervals as bases and with heights given by the values of the function at either endpoints of the subintervals.

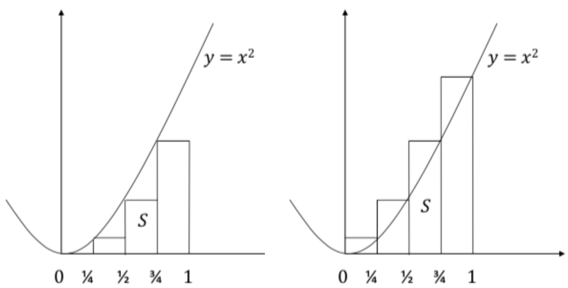


Figure 22: Using Rectangles for an Integral

Using the left endpoints of the subintervals, each rectangle has a width of $\frac{1}{4}$ and heights equal to $(0)^2, (\frac{1}{4})^2, (\frac{2}{4})^2, (\frac{3}{4})^2$. If we let L_4 be the sum of the areas of these approximating rectangles, we get

$$L_4 = \left(\frac{1}{4}\right)(0) + \left(\frac{1}{4}\right)\left(\frac{1}{16}\right) + \left(\frac{1}{4}\right)\left(\frac{4}{16}\right) + \left(\frac{1}{4}\right)\left(\frac{9}{16}\right) = 0.21875$$

Using the right endpoints of the subintervals, each rectangle has a width of $\frac{1}{4}$ again and heights equal to $(0)^2, (\frac{1}{4})^2, (\frac{2}{4})^2, (\frac{3}{4})^2$. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \left(\frac{1}{4}\right)\left(\frac{1}{16}\right) + \left(\frac{1}{4}\right)\left(\frac{4}{16}\right) + \left(\frac{1}{4}\right)\left(\frac{9}{16}\right) + \left(\frac{1}{4}\right)\left(\frac{16}{16}\right) = 0.46875$$

We note that the area A of S must be between L_4 and R_4; that is, $L_4 < A < R_4$ or $0.21875 < A < 0.46875$. We obtain better estimates by increasing the number of strips.The table below shows the result of similar calculations using n rectangles, whose heights are found with left endpoints L_n or right endpoints R_n. A good estimate is obtained by averaging the numbers.

n	L_n	R_n
10	0.285	0.385
20	0.309	0.359
30	0.317	0.350
50	0.323	0.343
100	0.328	0.338
1000	0.333	0.333

Notice that when the number of rectangles is increased ($n \to \infty$), the difference between the L_n and the R_n nears zero. In general, the width of the interval $[a, b]$ is $b - a$, so the equal width of each of the n rectangles is

$$\Delta x = \frac{b-a}{n}$$

The bases of the rectangles divide the interval into n subintervals, $[a, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, b]$. Instead of using the left or right endpoints, we could take the height of the ith rectangle to be the value of f at any number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, ..., x_n^*$ the **sample points**.

3.2.1 Reimann Sum

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{22}$$

We call the sum on the right hand side of the expression as the **Reimann sum**. It approximates the area under the curve from a to b. This approximation gets better and better as n increases.

3.2.2 Definite Integral

If f is a continuous function defined for $x \in [a, b]$, the definite integral of f from a to b is equal to equation 22 if the limit exists

$$\int_a^b f(x)\ \mathrm{dx} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{23}$$

When they say f is a Riemann-integrable or integrable, a and b are called the limits of integration. a is the lower limit and b is the upper limit.

Lemma 3.3. *If f is continuous, the limit in the definition always exists and gives the same value no matter how we choose the sample points. This implies that a function is integrable only if it is continuous. Therefore, continuity is a necessary condition for integrability*

3.2.3 Fundamental Theorems of Calculus

Let the function f be continous on the closed interval $[a, b]$ and let t be any number in $[a, b]$. If g is the function defined by

$$g(t) = \int_a^t f(x)\ \mathrm{dx}$$

Then it is reasonable to conclude that $g'(t) = f(t)$. Note that it should follow that g is also a continuous function on $[a, b]$.

Theorem 3.4. *Fundamental Theorem of Calculus 1: The derivative of an anti-derivative (i.e. the integral) is the original function itself*

If f is continous on $[a, b]$, then the expression below were F is any antiderivative of f is the definite integral.

$$\int_a^b f(x)\ \mathrm{dx} = F(b) - F(a)$$

Theorem 3.5. *If the antiderivative of the function being integrated is known, then the integral may be evaluated by subtracting the values of the antiderivative at the endpoints of the interval.*

We often use the notation below to denote this operation

$$F(b) - F(a) = F(x)|_a^b$$

For example, say you were asked to solve the following integral.

$$\int_1^2 (3x^2 - 4x)\ \mathrm{dx}$$

We first apply the factorization techniques

$$3\int_1^2 x^2\ \mathrm{dx} - 4\int_1^2 x\ \mathrm{dx}$$

We apply the rules of integration discussed previously and follow the operation of the definite integral

$$3\left(\frac{1}{3}x^3\Big|_1^2\right) - 4\left(\frac{1}{2}x^2\Big|_1^2\right)$$

Substitute the limits of integration using the order $F(b) - F(a)$.

$$3\left(\frac{(2)^3}{3} - \frac{(1)^3}{3}\right) - 4\left(\frac{(2)^2}{2} - \frac{(1)^2}{2}\right)$$

Simplifying

$$3\left(\frac{7}{3}\right) - 4\left(\frac{3}{2}\right) = 7 - 6 = 1$$

Therefore, the solution to our problem is 1 which is also under the area of the curve $f(x) = 3x^2 - 4x$ from the interval $[1, 2]$. Notice that we have no need for our constant/coefficient of integration. This is because we have a starting and ending limit represented by the bounds or limits of integration.

Evaluate the following definite integrals

1. $\int_{-2}^{2} x^3\ \mathrm{dx}$
2. $\int_{-1}^{1} e^x\ \mathrm{dx}$
3. $\int_0^1 (x - \sqrt{x})\ \mathrm{dx}$
4. $\int_0^1 (4 + 3x^2)\ \mathrm{dx}$
5. $\int_0^4 \sqrt{x}\ \mathrm{dx}$
6. $\int_1^3 \ln(x)\ \mathrm{dx}$
7. $\int_4^2 \left(\frac{u^2}{3} + 1\right)\ \mathrm{du}$

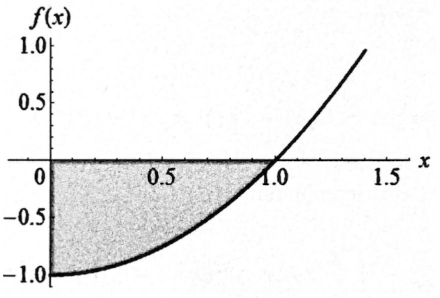


Figure 23: Showing a Shaded A of $f(x) = x^2 - 1$

For example, let us graph part of the function $f(x) = x^2 - 1$

Suppose you were asked to determine the shaded area in the graph. We can get this using a definite integral. Note that $f(x) < 0$ on $[0, 1]$. The (definite) integral of f from 0 to 1 is the shaded area.

$$\int_0^1 (x^2 - 1)\ \mathrm{dx} = \int_0^1 \left(\frac{1}{3}x^3 - x\right) = \frac{1}{3} - 1 = -\frac{2}{3}$$

The Fundamental Theorem of Calculus is not a procedure for finding indefinite integrals, but for evaluating definite integrals, given that we known an indefinite integral of the integrand.

3.2.4 Additional Rules on Definite Integrals

The rules from definite integration are largely the same as indefinite integration. However, there are a few additions or modifications to the existing rules. First, the constant function rule can be extended to accommodate the bounds

$$\int_a^b c\ \mathrm{dx} = c(b - a) \tag{24}$$

Taking the negative of the integral merely gives you the integral with the upper and lower bounds reversed

$$\int_a^b f(x)\ \mathrm{dx} = -\int_b^a f(x)\ \mathrm{dx} \tag{25}$$

Say there was a point c which is inside $[a, b]$. Then, the definite integral may be split into a simple sum for as long as the ends add up.

$$\int_a^b f(x)\ \mathrm{dx} = \int_a^c f(x)\ \mathrm{dx} + \int_v^b f(x)\ \mathrm{dx} \tag{26}$$

If $f(x) \geq 0$ for $x \in [a, b]$, then

$$\int_a^b f(x)\ \mathrm{dx} \geq 0$$

If $f(x) \geq g(x)$ for $x \in [a, b]$, then

$$\int_a^b f(x)\ \mathrm{dx} \geq \int_a^b g(x)\ \mathrm{dx}$$

If $m \leq f(x) \leq M$ for $x \in [a, b]$, then the form below holds where m and M are real numbers.

$$m(b - a) \leq \int_a^b f(x)\ \mathrm{dx} \leq M(b - a)$$

3.3 Integration by Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x) \text{ dx} = \int f(u) \text{ du} \tag{27}$$

Notice that if $u = g(x)$, then $du = g'(x)$ dx. Integration by substitution is used to evaluate integrals which cannot fully be solved using just the typical rules or techniques.

1. Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of the composite function.
2. Find $du = g'(x)$ dx
3. Use the substitution $u = g(x)$ and $du = g'(x)$ dx to convert the entire integral into one involving only u.
4. Evaluate the resulting integral
5. Replace u by $g(x)$ to obtain the final solution as a function of x.

For example, say we evaluate $\int (x^2 + 1)^5 2x$ dx.

Let $u = x^2 + 1$ or the "inner function". Then $du = 2x$ dx. Substituting into the integral, we get

$$\int (x^2 + 1)^5 2x \text{ dx} = \int u^5 \text{ du}$$

We can now apply the basic rules of integration

$$\int u^5 \text{ du} = \frac{u^6}{6} + C$$

Substituting the initial function back

$$\frac{(x^2 + 1)^6}{6} + C$$

Evaluate the following integrals

1. $\int (x^2 + 50x + 9)^{20}(2x + 50)$ dx
2. $\int 2(2x + 4)^5$ dx
3. $\int e^{x^2+1}$ dx
4. $\int e^{kx}$ dx, k is a constant
5. $\int_0^4 \sqrt{2x + 1}$ dx
6. $\int_0^2 (y^2 + 1)^3 2y$ dy

3.4 Integration by Parts

3.4.1 Integration by Parts of Indefinite Integrals

If $f(x)$ and $g(x)$ are differentiable functions that are integrable over $[a, b]$, then

$$\int f(x)g'(x) \text{ dx} = f(x)g(x) - \int g(x)f'(x) \text{ dx} \tag{28}$$

If we let $u = f(x)$ and $v = g(x)$, then $\text{du} = f'(x)$ dx and $\text{dv} = g'(x)$ dx. We can state the result above into the more conventional and well known integration by parts form.

$$\int u \text{ dv} = uv - \int v \text{ du} \tag{29}$$

The success of using integration by parts depends on the proper choice of u and dv. In general, choose u and dv such that

- du is simpler that u, and
- dv is easy to integrate

For example, suppose we evaluate

$$\int xe^x \text{ dx}$$

We let $u = x$ and dv= e^x dx. This means that du=dx and $v = \int e^x \text{ dx} = e^x$. Using the integration by parts formula, we get

$$\int u \text{ dv} = uv - \int v \text{ du}$$
$$\int xe^x \text{ dx} = xe^x - \int e^x \text{ dx}$$
$$xe^x - e^x + C$$

Evaluate the following indefinite integrals

1. $\int xe^{2x}$ dx
2. $\int x^2 \ln(x)$ dx
3. $\int xe^x$ dx
4. $\int x\sqrt{x}$ dx

3.4.2 Integration by Parts for Definite Integrals

If $f(x)$ and $g(x)$ are differentiable functions that are integrable over $[a, b]$, then

$$\int_a^b f(x)g'(x) \text{ dx} = [f(b)g(b) - f(a)g(a)] - \int_a^b g(x)f'(x) \text{ dx}$$

Let $u = f(x)$ and $v = g(x)$, then du= $f'(x)$ dx and dv= $g'(x)$ dx, we can state the result as

$$\int_a^b u \text{ dv} = uv|_a^b - \int_a^b v \text{ du} \tag{30}$$

Alternatively, you can solve the indefinite integral first and apply the limits of integration afterwards. For example, recall our previous derivation for the integral of xe^x

$$\int xe^x \text{ dx} = xe^x - \int e^x \text{ dx} = xe^x - e^x$$

Applying the rules of definite integration

$$e^x(x-1)$$
$$e^2(2-1) - e^1(1-1) = e^2 \approx 7.39$$

3.5 Integration by Using Partial Fractions

Consider the integral

$$\int \frac{2x+1}{x^2-2x-3} \text{ dx}$$

When the integrand is a rational function like the one above, integration may be performed by first decomposing it into the sum of simpler rational functions called **partial fractions**. These fractions can then be integrated using typical methods. The method of decomposing a rational function is straightforward when the *degree of the numerator is less than the degree of the denominator.*

For example, let us use the integral above which is $\int \frac{2x+1}{x^2-2x-3}$ dx. To evaluate this integral, we first have to *factor the denominator*

$$x^2 - 2x - 3 = (x-3)(x+1)$$

We then set the original function be a sum of two partial fractions

$$\frac{2x+10}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$

Multiplying both sizes by $x^2 - 2x - 3$ to cancel out all the denominators. Note that $x^2 - 2x - 3 = (x-3)(x+1)$

$$2x + 10 = A(x+1) + B(x-3)$$

We then look for values of x that will eliminate the terms involving A and then B. This will enable us to solve for the unknowns A and B. In this case, we have $x = -1$ to eliminate A and $x = 3$ which will eliminate B.

When $x = -1$, we get

$$-2 + 10 = A(0) + B(-4); \text{ hence, } B = -2$$

When $x = 3$, we get

$$6 + 10 = A(4) + B(0); \text{ hence, } A = 4$$

Therefore

$$\frac{2x+10}{x^2-2x-3} = \frac{4}{x-3} + \frac{-2}{x+1}$$

We can now perform the integration

$$\int \frac{2x+10}{x^2-2x-3} \text{ dx} = \int \frac{4}{x-3} \text{ dx} + \int \frac{-2}{x+1} \text{ dx}$$

$$4\int \frac{1}{x-3} \text{ dx} - 2\int \frac{1}{x+1} \text{ dx}$$

$$4\ln(x-3) - 2\ln(x+1) + C$$

Evaluate the following indefinite integrals

1. $\int \frac{x+2}{x^2-2x}$ dx
2. $\int \frac{3}{x^2-25}$ dx
3. $\int \frac{x^2+4}{x^3-3x^2+2x}$ dx

3.6 Integration by Using Partial Fractions with Repeated Factors

A rational function whose denominator has repeated factors $(x-a)^k$ must be decomposed into the sum of k partial fractions, one for each power of $(x-a)$. After decomposing, the fractions can then be integrated using typical methods. Again, this method of decomposing a rational function is easy to use when the degree of the numerator is less than the degree of the denominator.

For example, say you were given the integral

$$\int \frac{3x+5}{x^3-2x^2+x} \text{ dx}$$

We can rewrite the integrand as

$$\frac{3x+5}{x^3-2x^2+x} = \frac{3x+5}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiplying both sides of the equation by $x(x-1)^2$, we get

$$3x+5 = A(x-1)^2 + Bx(x-1) + C(x)$$

We now find values of x that can cancel out some of the letters. We have $x=1$ which will eliminate A and B and $x=0$ which will eliminate B and C.

When $x=1$, we get

$$3+5 = A(0) + B(0) + C(1); \text{ hence } C = 8$$

When $x=0$, we get

$$5 = A(1) + B(0) + C(0); \text{ hence } A = 5$$

Since A and C are already known, we may substitute any number to solve for B, say $x=2$.

$$3(2)+5 = 5(1) + B(2)(1) + 8(2); \text{ hence } B = -5$$

Therefore

$$\frac{3x+5}{x^3-2x^2+x} = \frac{3x+5}{x(x-1)^2} = \frac{5}{x} + \frac{-5}{x-1} + \frac{8}{(x-1)^2}$$

We can now perform the integration

$$\int \frac{3x+5}{x^3-2x^2+x} \text{ dx} = \int \frac{5}{x} \text{ dx} + \int \frac{-5}{x-1} \text{ dx} + \int \frac{8}{(x-1)^2} \text{ dx}$$

$$5\ln(x) - 5\ln(x-1) - \frac{8}{x-1} + C$$

3.7 Improper Integrals

Integrals of the form below are called **improper integrals**

$$\int_a^{\infty} f(x)\ \mathrm{dx}$$

$$\int_{-\infty}^{b} f(x)\ \mathrm{dx}$$

$$\int_{-\infty}^{+\infty} f(x)\ \mathrm{dx}$$

Another type of improper integral is one where the limits of integration a and b are finite but f has an infinite discontinuity on the interval $[a, b]$. For example, the integrals below are improper because for the first, the integrand goes to infinity as x approaches zero, an interior point of $[-1, 1]$, while for the second, the integrand goes to infinity as x approaches zero, the lower limit of integration.

$$\int_{-1}^{1} \frac{1}{x}\ \mathrm{dx}$$

$$\int_{0}^{1} \frac{1}{x^2}\ \mathrm{dx}$$

Theorem 3.6. *If $\int_a^t f(x)\ dx$ exists for ever number $t \geq a$, then $\int_a^{\infty} f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$ provided this limit exists*

The theorem basically states that an solution exists for an improper integral with an upper limit of infinity for as long as the limit exists.

Theorem 3.7. *If $\int_t^b f(x)\ dx$ exists for ever number $t \leq b$, then $\int_{-\infty}^{b} f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$ provided this limit exists*

The theorem basically states that an solution exists for an improper integral with an lower limit of infinity for as long as the limit exists.

The **improper integrals** $\int_a^{\infty} f(x)$ dx and $\int_{-\infty}^{b} f(x)$ dx are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

If both $\int_a^{\infty} f(x)$ dx and $\int_{-\infty}^{b} f(x)$ dx are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{dx} = \int_{-\infty}^{a} f(x)\ \mathrm{dx} + \int_a^{\infty} f(x)\ \mathrm{dx}$$

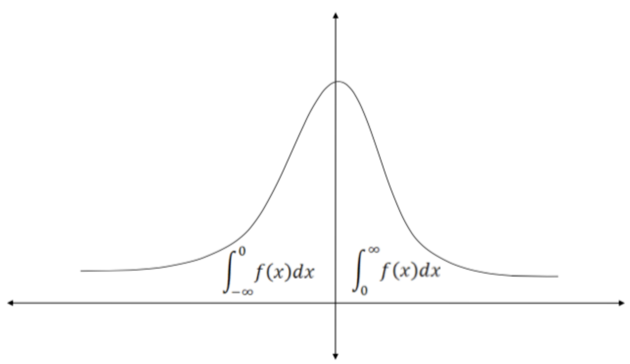


Figure 24: Showing $\int_{-\infty}^{\infty} f(x)$ dx

For example, determine the convergence or divergence of the following integrals.

Say we have $\int_1^\infty \frac{1}{x}$ dx

$$\int_1^\infty \frac{1}{x} \text{ dx} = \lim_{t\to\infty} \int_1^t \frac{1}{x} \text{ dx} = \lim_{t\to\infty} \ln(x)|_1^t = \lim_{t\to\infty} [\ln(t) - \ln(1)] = \infty$$

This given improper integral is **divergent** because the limit does not exist.

Say we have $\int_{-\infty}^0 e^x$ dx

$$\int_{-\infty}^0 e^x \text{ dx} = \lim_{t\to-\infty} \int_t^0 e^x \text{ dx} = \lim_{t\to-\infty} e^x|_t^0 = \lim_{t\to-\infty} [e^0 - e^t] = 1$$

This given improper integral is **convergent** because the limit exists.

Say we were given with $\int_{-\infty}^\infty 2xe^{-x^2}$ dx. By definition, we have

$$\int_{-\infty}^\infty 2xe^{-x^2} \text{ dx} = \int_{-\infty}^0 2xe^{-x^2} + \int_0^\infty 2xe^{-x^2} \text{ dx dx}$$

We evaluate the integrals on the right-hand side term by term. To do so, we have to use integration by substitution. Let $u = -x^2$ and du $= -2x$ dx

$$\int_{-\infty}^0 2xe^{-x^2} \text{ dx} = \lim_{t\to-\infty} \int_t^0 2xe^{-x^2} \text{ dx} = \lim_{t\to-\infty} \int_t^0 -e^u \text{ du} = \lim_{t\to-\infty} -e^{-x^2}|_t^0 = -e^0 - (-e^{-t^2}) = -1$$

$$\int_0^\infty 2xe^{-x^2} \text{ dx} = \lim_{t\to\infty} \int_0^t 2xe^{-x^2} \text{ dx} = \lim_{t\to\infty} \int_0^t -e^u \text{ du} = \lim_{t\to\infty} -e^{-x^2}|_0^t = -e^{-t^2} - (-e^0) = 1$$

Thus, $\int_{-\infty}^\infty 2xe^{-x^2}$ dx $= -1 + 1 = 0$, which means that the given integral is **convergent**. The reason the integral is zero is that the integral from negative infinity to zero is a negative area while the integral from zero to positive infinity is a positive area. Those areas are equal as seen below.

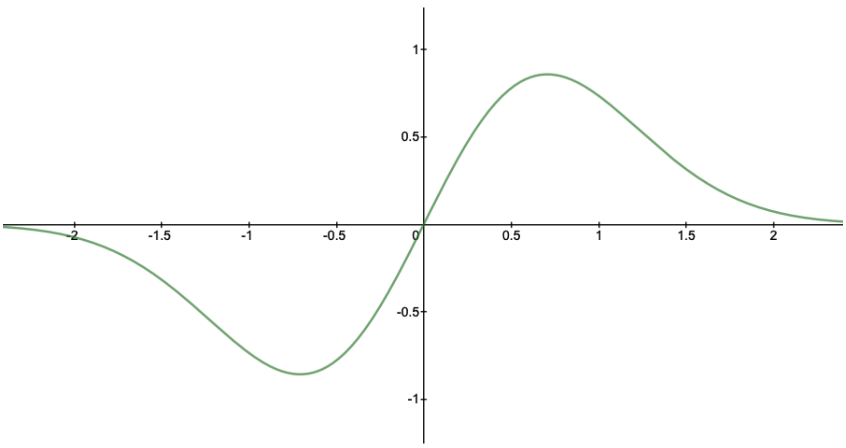


Figure 25: Showing $2xe^{-x^2}$

Determine the convergence or divergence of the following improper integrals

1. $\int_1^\infty \frac{1}{x^2}$ dx
2. $\int_0^\infty e^{-x}$ dx

Theorem 3.8. *If f is continuous on $(a, b]$ and is discontinuous at a, then $\int_a^b f(x)\ dx = \lim_{t \to a} \int_t^b f(x)\ dx$ provided that the limit exists*

The theorem just states that if the lower limit of integration is a point of discontinuity, the integral may still be evaluated for as long as the limit exists.

Theorem 3.9. *If f is continuous on $(a, b]$ and is discontinuous at b, then $\int_a^b f(x)\ dx = \lim_{t \to b} \int_a^t f(x)\ dx$ provided that the limit exists*

The theorem just states that if the upper limit of integration is a point of discontinuity, the integral may still be evaluated for as long as the limit exists.

The **improper integral** $\int_a^b f(x)$ dx whether discontinuous at a or b is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

If f has a discontinuity at c, where $a < c < b$, then we define

$$\int_a^b f(x)\ \mathrm{dx} = \int_a^c f(x)\ \mathrm{dx} + \int_c^b f(x)\ \mathrm{dx}$$

Determine the convergence or divergence of the following improper integrals

1. $\int_0^1 \frac{1}{\sqrt{x}}$ dx
2. $\int_0^3 \frac{1}{\sqrt{3-x}}$ dx
3. $\int_{-2}^3 \frac{1}{x^3}$ dx

3.8 Economic Applications of Integrations

3.8.1 Finding Total Cost Functions from Marginal Cost Functions

Recall that in Economics, the derivative $F'(x)$ of a function $F(x)$ is called a **marginal function**. $F(x)$ is then called a **total function** Thus, a total function is simply the integral of a marginal function.

Given a total cost function $C(q)$, the process of differentiation can yield the marginal cost function $C'(q)$. Because integration is merely the inverse operation of differentiation, integrating the marginal cost function $C'(q)$ should yield the total cost function $C(q)$.

For example, the marginal cost function of a firm is given by $MC(q) = 3q^2 - 30q + 80$. The fixed cost is known to be 100. Find the total cost function. To do this, we merely integrate the marginal cost function and add the fixed cost.

$$C(q) = \int (3q^2 - 30q + 80)\ \mathrm{dq} = 3\int q^2\ \mathrm{dq} - 30\int q\ \mathrm{dq} + 80\int\ \mathrm{dq} = q^3 - 15q^2 + 80q + 100$$

Hence, the total cost function is $q^3 - 15q^2 + 80q + 100$

3.8.2 Finding Total Revenue Functions from Marginal Revenue Functions

By integrating the **marginal revenue** function $R'(q)$, we can find the **total revenue function** $R(q)$. But total revenue R is also defined by the equation $R = pq$, where p is the price per unit and q is the total number of output produced. Thus, we have $p = R/q$ or $p(q) = R(q)/q$. This means that we can get the demand function $p(q)$ once we have the total revenue function $R(q)$. Because it is generally true that the total revenue is zero when $q = 0$, this fact can be used to evaluate the arbitrary constant when finding the total revenue function from the marginal revenue function.

For example, say the marginal revenue function is given by $MR(q) = 8 - 6q - 2q^2$ where q denotes output produced. Find the total revenue function R. To get R, we just simply integrate the marginal revenue function.

$$R(q) = \int (8 - 6q - 2q^2)\ \mathrm{dq} = 9q - 3q^2 - \frac{2}{3}q^3 + C$$

Because $R(0) = 0$, we get $C = 0$. Hence, the total revenue function is

$$R(q) = 8q - 3q^2 - \frac{2}{3}q^3$$

3.8.3 National Income, Consumption, and Savings

Let $C(Y)$ denote an economy's consumption function where C is the household consumption and Y is the national income. Then the **marginal propensity to consume** (MPC) is given by $C'(Y)$, and the consumption function $C(Y)$ is just the integral of the MPC with respect to Y. In addition, since **private savings** S is defined by the equation $S(Y) = Y - C(Y)$, then the **marginal propensity to save (MPS)** is given by $S'(Y) = 1 - C'(Y)$.

For example, suppose the marginal propensity to save is given by the function $S'(Y) = 0.3 - 0.1Y^{-0.5}$, and if private savings S is zero when national income Y is Php 81, then

$$S(Y) = \int (0.3 - 0.1Y^{-0.5})\ \mathrm{dY} = 0.3Y - 0.2Y^{0.5} + k$$

The specific value of k can be found from the fact that $S = 0$ when $Y = 81$. Substitution of this information into the preceding integral will yield the value for the arbitrary constant k.

Thus we have

$$0 = 0.3(81) - 0.2(81)^{0.5} + k$$
$$k = -22.5$$

Therefore, the private savings function is $S(Y) = 0.3Y - 0.2Y^{0.5} - 22.5$ while the consumption function is obtained from

$$S(Y) = Y - C(Y)$$
$$0.3Y - 0.2Y^{0.5} - 22.5 = Y - C(Y)$$
$$C(Y) = 0.7Y + 0.2Y^{0.5} + 22.5$$

3.8.4 Investment, Capital Formation, and Capital Accumulation

The process of adding to a given stock of capital is referred to as **capital formation**. If this process is considered to be continuous over time, capital stock K can be expressed as a function of time $K(t)$, and the rate of capital formation is then given by $\frac{dK}{dt} = K'(t)$. The rate of capital formation at time t is the same as the rate of net investment flow at time t, denoted by $I(t)$.

$$K(t) = \int K'(t)\ \mathrm{dt} = \int I(t)\ \mathrm{dt}$$

In the form above, an initial condition $K(0)$ must be specified to evaluate the arbitrary constant of integration.

Suppose that the net investment flow is described by $I(t) = 3\sqrt{t}$ and that the initial capital stock at time $t = 0$ is $K(0) = K_0$. By integrating $I(t)$ with respect to t, we obtain

$$K(t) = \int 3\sqrt{t}\ \mathrm{dt} = 2t^{1.5} + C$$

Next, letting $t = 0$, we find $K(0) = C$ or $K_0 = C$. Therefore, the *time path of* K is

$$K(t) = 2t^{1.5} + K_0$$

Sometimes, the concept of gross investment is used together with net investment in a model. Denoting gross investment by I_g and net investment by I, we can relate them to each other by the equation $I_g = I + \delta K$, where δ represents the *rate of depreciation of capital* and δK is the rate of *replacement investment*

Since $K(t)$ is the antiderivative or integral of $I(t)$, we may write the definite integral

$$\int_a^b I(t)\ \mathrm{dt} = K(b) - K(a)$$

This indicates the **total capital accumulation** during the time interval $[a, b]$. Of course, this also represents the are under the $I(t)$ curve.

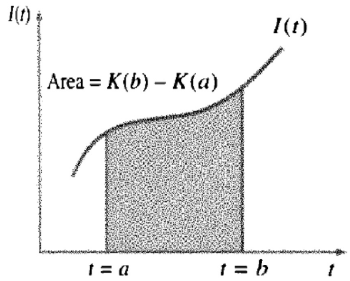


Figure 26: Showing Capital Accumulation

For example, if net investments, in thousands per year, is a non-constant flow that is determined by $I(t) = 3\sqrt{t}$, then the capital accumulated during the time interval $[0, 4]$ lies in the definite interval.

$$\int_0^4 3\sqrt{t}\ \mathrm{dt} = (2t^{1.5})|_0^4$$

$$(2)(4)^{1.5} - (2)(1)^{1.5} = 16 - 2 = 14$$

We may express the amount of capital accumulation during the time interval $[0, t]$, for any investment rate $I(t)$, by the definite integral

$$K(t) - K(0) = \int_0^t I(t)\ \mathrm{dt}$$

$$K(t) = K(0) + \int_0^t I(t)\ \mathrm{dt}$$

The preceding equation yields the following expression for the time path $K(t)$. That is, the amount of K at any time t is the initial capital plus the total capital accumulation that has occurred since.

For example, suppose that a firm begins at time $t = 0$ with capital stock of $K(0) = 500,000$ and, in addition to replacing any depreciated capital, is planning to invest in new capital at the rate $I(t) = 600t^2$. The planned level of capital stock t years from now is computed according to

$$K(t) = K(0) + \int_0^t I(t)\ \mathrm{dt}$$

$$K(t) = 500,000 + \int_0^t 600t^2 \text{ dt}$$

If $t = 10$, then capital stock 10 years from now is $K(10) = 700,000$.

3.8.5 Consumer, Producer, and Social Surpluses

The *Consumer's Surplus (CS)* represents the welfare of consumers. It is the extra amount consumers would have been willing to pay but did not. Thus, CS is calculated as a definite integral:

$$CS = \int_0^{Q^*} D(Q) \text{ dQ} - p^*Q^* \quad (31)$$

In the equation above, $D(Q)$ is the inverse market demand function, p^* is the market equilibrium price, and Q^* is the market equilibrium output.

On the other hand, the *Producer's Surplus (PS)* represents the welfare of producers or sellers. It is the extra amountvproducers received above what they were willing to accept. Thus, PS is calculated as a definite integral:

$$PS = p^*Q^* - \int_0^{Q^*} S(Q) \text{ dQ} \quad (32)$$

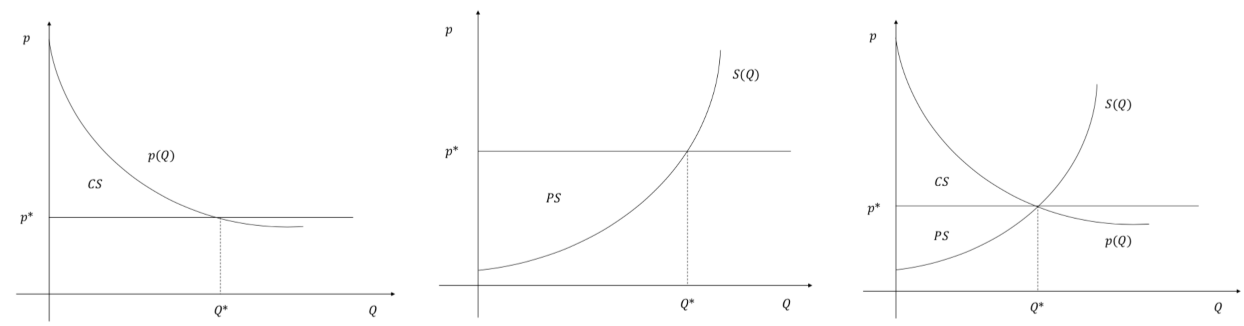
In the equation above, $S(Q)$ is the market supply function, p^* is the market equilibrium price, and Q^* is the market equilibrium output.

Lastly, the *Social Surplus (SS)* (or economic surplus or net market surplus) is the total welfare of both consumer and producer. It is the sum of both consumer's and producer's surplus; that is $CS + PS$ or

$$SS = CS + PS = \int_0^{Q^*} D(Q) \text{ dQ} - p^*Q^* + \left(p^*Q^* - \int_0^{Q^*} S(Q) \text{ dQ}\right)$$

Simplifying, we get

$$SS = \int_0^{Q^*} D(Q) \text{ dQ} - \int_0^{Q^*} S(Q) \text{ dQ} = \int_0^{Q^*} [D(Q) - S(Q)] \text{ dQ} \quad (33)$$



(a) Illustrating Consumer Surplus (b) Illustrating Producer Surplus (c) Illustrating Social Surplus

Figure 27: Illustrating the Economic Surpluses

For example, given the respective inverse market demand and supply functions $P = 50 - 0.1Q$ and $P = 0.2Q + 20$, find the consumer's, producer's, and social surpluses. To start off, we need to compute for the equilibrium quantity and price. We know that at equilibrium, supply is equal to demand. We need to equate the two functions we were given with and compute for Q.

$$50 - 0.1Q = 0.2Q + 20$$
$$30 = 0.3Q$$
$$Q^* = 100$$

Knowing that $Q^* = 100$, we can plug this into either the inverse demand or inverse supply to get the market equilibrium price. Say we plug it into the inverse demand function

$$P = 50 - 0.1(100) = 40$$
$$P^* = 40$$

We now know that $P^* = 40$ and $Q^* = 100$, we can now use these information in computing for the economic surpluses. We start with the consumer surplus.

$$CS = \int_0^{Q^*} D(Q)\ \mathrm{dQ} - p^*Q^* = \int_0^{100} (50 - 0.1Q)\ \mathrm{dq} - (100)(40)$$
$$50\int_0^{100}\ \mathrm{dq} - 0.1\int_0^{100} Q\ \mathrm{dq} - 4000$$
$$50(Q)|_0^{100} - 0.1\left(\frac{Q^2}{2}\right)\Big|_0^{100} - 4000$$
$$50(100-0) - 0.1\left(\frac{100^2}{2} - \frac{0^2}{2}\right) - 4000$$
$$5000 - 500 - 4000$$
$$CS = 500$$

Using the same information to obtain the producer surplus

$$PS = p^*Q^* - \int_0^{Q^*} S(Q)\ \mathrm{dQ} = (100)(40) - \int_0^{100} (20 + 0.2Q)\ \mathrm{dq}$$
$$4000 - 20\int_0^{100}\ \mathrm{dq} - 0.2\int_0^{100} Q\ \mathrm{dq}$$
$$4000 - 20(Q)|_0^{100} - 0.2\left(\frac{Q^2}{2}\right)\Big|_0^{100}$$
$$4000 - 20(100-0) - 0.2\left(\frac{100^2}{2} - \frac{0^2}{2}\right)$$
$$4000 - 1000 - 2000$$
$$PS = 1000$$

Adding both the consumer and producer surpluses yields us the social surplus which is $SS = 1000 + 500 = 1500$. Alternatively, as an exercise, you can verify using the formula above.

Find the equilibrium point of the following demand and supply functions and calculate the social surplus at equilibrium

$$p(q) = D(q) = -\frac{q^2}{100} - x + 50$$
$$p(q) = S(q) = \frac{q^2}{25} + 10$$

4 Matrix Algebra

4.1 Introduction to Matrices

Partial equilibrium analysis can be relatively tedious to solve via the methods of elimination or substitution, especially when the system contains many equations. Fortunately, through the use of matrices, we can simplify the process of computation and even expand it up to n number of goods.

A **matrix** is a rectangular array A of numbers (or symbols representing numbers) and variables and is written in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The numbers in the array A are called *entries*. Each entry has double subscripts which indicate the address of the entry, i.e. a_{ij} is located in the ith row and jth column of A. A matrix with m rows and n columns is referred to as an (read m by n) matrix and m and n are called the **dimensions** of the matrix. If $m = n$, then the matrix is called a **square matrix** of order n.

4.1.1 Special Forms of Matrices

A **vector** is a matrix with either one row, called a *row matrix*, or one column, called a *column matrix*

$$x = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \end{pmatrix}$$

$$y = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix}$$

A **scalar** is a constant. In matrix formulation, this is a matrix with only one row and one column.

$$A = \begin{pmatrix} a_{11} \end{pmatrix}; \text{ or simply} A = (a)$$

A **square matrix** is a matrix that has the same number of rows and columns (i.e. $m = n$)

A **diagonal matrix** is a square matrix having non-zero elements only in the principal diagonal, i.e., the diagonal running from the upper left to the lower right.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

Note: All diagonal matrices are square matrices, but not all square matrices are diagonal matrices.

An **Identity Matrix** (or *unit matrix*) is a diagonal matrix that has values of 1 in its principal diagonal. Since it is a diagonal matrix, all non-diagonal entries must be zero.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note: A matrix pre or post multiplied by a conformable identity matrix is equal to the matrix itself, i.e. an identity matrix plays the same roles as 1 in scalar algebra.

A **Null Matrix** is a matrix that has 0's for its entries; plays a similar role to 0 in scalar algebra.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

An **Idempotent Matrix** is when say a matrix A multiplied to itself yields itself. That is, $A \cdot A = A$. The simplest examples of idempotent matrices are the identity matrix and the null matrix.

A **Triangular Matrix** is a matrix with non-zero elements in either the northeast or southwest of the principal diagonal while the rest of the elements are zero. We generally subclassify these into two.

- *Upper Triangular Matrix* - a matrix for which every diagonal entry in the northeast, including the principal diagonal, is non-zero

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

- *Lower Triangular Matrix* - a matrix for which every diagonal entry in the southwest, including the principal diagonal, is non-zero

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

A **Symmetric Matrix** is a square matrix for which the off-diagonal elements in the northeast are a mirror image of the off-diagonal elements in the southwest.

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{pmatrix}$$

There are **Equal Matrices** which are matrices with the same dimensions and the same corresponding entries in corresponding positions.

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{pmatrix}$$

A **Scalar Matrix** is a diagonal matrix where all the entris in the principal diagonal re equal. The identity matrix is an example of a scalar matrix.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The **Transpose Matrix** is the matrix which contains the entries for which the rows and columns of the matrix are interchanged. We denote the transpose of matrix A as A' or A^T

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \to A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

There are a couple of properties that transpose matrices have

1. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix

2. $(A^T)^T = A$ The transpose of a transpose is the original matrix
3. $(A \pm B)^T = A^T \pm B^T$
4. $(cA)^T = cA^T$
5. $(AB)^T = B^T A^T$
6. If A is symmetric, then $A = A^T$

4.2 Matrix Operations

4.2.1 Matrix Addition and Subtraction

Matrix addition and subtraction requires matrices to be conformable, i.e. matrices must have the same dimensions. For example, say Matrix A has a dimension of 2×3 and Matrix B has a dimension of 2×3. We can add and subtract matrix A and B since both matrices have the same dimension. Say we have a Matrix F which has a dimension of 3×3, we cannot add and subtract this to A or B as they are of different dimensions.

If matrices A and B are conformable, then matrix $C = A + B$ will also have the same dimension as A and B. The entries in matrix C are obtained by simply adding (subtracting) the entries of A and B with the same address. For example, say $A = \begin{pmatrix} 2 & 5 & 0 \\ 3 & -1 & 8 \end{pmatrix}$; $B = \begin{pmatrix} 6 & 1 & 2 \\ 7 & 9 & 0 \end{pmatrix}$

$$A + B = \begin{pmatrix} 2 & 5 & 0 \\ 3 & -1 & 8 \end{pmatrix} + \begin{pmatrix} 6 & 1 & 2 \\ 7 & 9 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 6 & 2 \\ 10 & 8 & 0 \end{pmatrix}$$

4.2.2 Scalar Multiplication

Scalar multiplication involves taking the product of a matrix and a scalar. To do so, simply multiply each entry of the matrix by the scalar. For example, say q is the scalar and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then:

$$qA = \begin{pmatrix} qa & qb \\ qc & qd \end{pmatrix}$$

4.2.3 Matrix Multiplication

Matrix multiplication is not as simple as scalar multiplication since (1) it requires conformity between the two matrices and (2) involves calculation of the matrices' inner product.

For $A \cdot B = C$, A is called the *lead matrix*, and B is called the *lag matrix*. You call this as "pre-multiplying matrix B by matrix A" or "post-multiplying matrix A by matrix B".

Theorem 4.1. *The number of columns in the lead matrix must be equal to the number of rows in the lag matrix. (Conformity Condition)*

For example, say matrix A is an $(m \times n)$ matrix, matrix B is an $(n \times k)$. By the conformity condition, $A \cdot B$ is possible since A has n columns and B has n rows. By contrast, $B \cdot A$ is not possible since it doesn't meet the conformity condition.

Dimensions of the product of two matrices would be the "outer product" $A\dot{B} = C$ where C has the same number of rows as the lead matrix and the same number of columns as the lag matrix. To illustrate, each element in C has an inner product of two vectors in matrices A and B. To illustrate the concept of the inner product, consider vectors u and v below

$$u = \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix}; \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Note that $u \cdot v$ satisfies the conformity condition. The inner product of these two are given by the scalar $[uv] = [u_1 v_1 + \ldots + u_n v_n]$ For example, take the case below

$$u \cdot v = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix} = [1 \cdot 5 + 2 \cdot 10 + 3 \cdot 15] = [5 + 20 + 45] = [70]$$

In general, for $A \cdot B = C$, we have

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \ldots & b_{1k} \\ \vdots & \ddots & \ldots \\ a_{n1} & \ldots & a_{nk} \end{pmatrix} = \begin{pmatrix} c_{11} & \ldots & c_{1k} \\ \vdots & \ddots & \ldots \\ c_{m1} & \ldots & c_{mk} \end{pmatrix}$$

Note that A is $m \times n$, B is $n \times k$, and as a result of the multiplication process, C is $m \times k$.

c_{ij} is the entry of matrix C that appears in the ith row and jth column. It is also the inner product of the ith row of A and the jth column of B.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}$$

Note that

$$c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21}$$

$$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21}$$

For example, say $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$. Perform $A \cdot B$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 1 \cdot 10 + 2 \cdot 20 \\ 3 \cdot 10 + 4 \cdot 20 \end{pmatrix} = \begin{pmatrix} 50 \\ 110 \end{pmatrix}$$

Perform the following indicated matrix operations

1. $A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{pmatrix}$; $b = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$, $A \cdot b$

2. $A = \begin{pmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{pmatrix}$, $\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A \cdot \mathrm{x}$

3. $M = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 5 & 6 & 2 & 7 \\ 7 & 6 & 5 & 2 \\ 1 & 2 & 3 & 5 \end{pmatrix}$, $U = \begin{pmatrix} 5 & 6 \\ 1 & 3 \\ 5 & 6 \\ 2 & 3 \end{pmatrix}$, $M \cdot$

4. Consider three consumers ($i = 1, 2, 3$) who purchased four goods ($j = 1, 2, 3, 4$). Let a_{ij} denote the quantity of good j purchased by consumer i at price p_j. The purchases may be arranged in a matrix A and the prices in a vector p

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

What is Ap equal to? What does each entry of Ap represent?

While matrices, like numbers, can undergo the operations of addition, subtraction, and multiplication – subject to the conformability conditions – it is not possible to divide one matrix by another. That is, we cannot write A/B. However, instead of writing A/B, we can instead write and solve for AB^{-1} where B^{-1} is the inverse of matrix$B$. But take note that the product of AB^{-1} may not be necessarily the same as the product of $B^{-1}A$. Therefore, the expression A/B cannot be used without causing confusion and must, thus, be avoided.

4.2.4 Properties of Matrix Operations

1. **Commutative**. Matrix addition is commutative, $A + B = B + A$, for as along as A and B are conformable for addition. Scalar multiplication is also commutative, that is, $kA = Ak$, where k is some scalar. Matrix multiplication is not commutative, that is, $A \cdot B \neq B \cdot A$. Even if $A \cdot B$ and $B \cdot A$ are possible, $A \cdot B = B \cdot A$ is not necessarily true.

2. **Associative**. Matrix addition is associative $(A + B) + C = A + (B + C)$, for as long as A and B are conformable for addition. Matrix multiplication is associative $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, as long as the conformability condition is satisfied by each pair of adjacent matrices.

3. **Distributive**. Matrix multiplication is distributive $A \cdot (B + C) = A \cdot B + A \cdot C$

4. Given scalars c, k, and matrices A, B
   - $(c + k)A = cA + kA$
   - $(ck)A = c(kA) = k(cA)$
   - $c(AB) = (cA)B = A(cB)$
   - $c(A + B) = cA + cB$

4.3 Solving Systems of Linear Equations using Matrices

4.3.1 Matrix Algebra and the Matrix Form

Matrix algebra provides a compact way of writing an equation system. Once the system has been written down, matrix algebra also provides a mechanism for testing the existence of a solution to an equation system. It is an alternative procedure for finding the solution to any linear equation system. Note that matrix algebra applies to *linear equations* only.

Matrices allow us to write equations in a compact format. The process involves identifying the endogenous variables, coefficients, and constants (along with exogenous variables), and organizing these into separate matrices. The equation system with m equations and endogenous variables in expanded form may be written as

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m = d_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mm}x_m = d_m$$

Given this system of equations

1. You have m equations and m endogenous variables, i.e. $x_1, x_2, ..., x_m$

2. On the left hand side, you have $m \times m$ a_{ij}'s which are the coefficients of the endogenous variables.

3. On the right hand side, you have $m \times 1$ d's, which are the exogenous variables or constant terms.

Thus, given this system of equations, we can make a matrix of coefficients, endogenous variables, and exogenous variables such that

$$Ax = d \tag{34}$$

In this equation, we have A which is an $m \times m$, x which is an $m \times 1$ and d which is an $m \times 1$ matrix. Putting these together we have

$$\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \dots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}$$

This system of equations written in matrix form can now be solved to obtain the solution values of the endogenous variables.

Rewrite the following systems of linear equations into matrix form

1.
$$x - 2y + 3z = 7$$
$$2x + y + z = 4$$
$$-3x + 2y - 2z = -10$$

2.
$$Q_d = 500 - 50P$$
$$Q_s = 50 + 25P$$
$$Q_d = Q_s$$

3.
$$C = 260 + 0.4Y$$
$$I = 300 + 0.5Y$$
$$Y = C + I + 110$$

4.3.2 Defining Determinants

For a square matrix, one can capture important information about the matrix in just a single number called the **determinant**. The determinant is useful in solving systems of linear equations.Necessary and sufficient conditions for a determinant to exist

- *Necessary Condition.* The matrix must be square, i.e. when transforming systems of linear equations into matrix form, the matrix will only be square if and only if **the number of equations is equal to the number of unknowns**.
- *Sufficient Condition.* All rows and columns of a square matrix must be **linearly independent** of each other. Linear independence prevents systems of equations from reducing into a single equation and, thus, ensures that we get a unique solution.

To illustrate linear independence

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \dots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

In the formulation above, v_i is a row vector containing the elements of row $i = (a_{i1} \ \dots \ a_{im})$. Two rows ($v_i$ and v_j) are *linearly dependent* if there exists a non-zero scalar k that satisfies $v_i + kv_j = 0 \; \forall \; i \neq j$

For example, let $v_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 & 4 & 6 \end{pmatrix}$. What value of k will satisfy the condition $v_i + kv_j = 0 \; \forall \; i \neq j$?

For example, if the coefficient matrix of a system of equations is

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

We find that since $(6 \quad 8 \quad 10) = 2(3 \quad 4 \quad 5)$, we have $v_3 = 2v_1 = 2v_1 + 0v_2$. Thus, the third row is expressible as a linear combination of the first two rows, and the rows are *not linearly independent*. Alternatively, we may write the previous equation as $2v_1 + 0v_2 - v_3 = 0$. Therefore, we find the scalars $k = 2, 0, -1$ such that $kv_i + kv_j + kv_l = 0$, respectively. Therefore, the three rows are not linearly independent.

Theorem 4.2. *A determinant would exist if and only if all rows and columns of the coefficient matrix are linearly independent*

Again, the third row in the previous matrix is only a multiple of the first row. Such linear dependence will reduce, in effect, the three rows into only two rows with three endogenous variables, which will yield an infinite number of solutions. Therefore, **linear independence** is a *sufficient condition* for a determinant and, thus, for a unique solution to exist. Given a small matrix, row or column independence is easily verified by visual inspection. However, for a matrix of larger dimension, visual inspection may not be feasible. Fortunately, we know that a determinant would exist if and only if all rows and columns of the coefficient matrix are linearly independent. Thus, to check for linear independence, we can simply solve for the determinant and see if it exists.

4.3.3 Solving for Determinants

The determinant of a matrix A is denoted by $|A|$ or $det(A)$.

For a 2×2 case, the determinant is the product of the principal diagonal elements less the product of the two off-diagonal elements. Given a general 2×2 matrix, the determinant is given as $|A|$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21} \tag{35}$$

For example, find the determinant of B, $|B|$

$$B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

$$|B| = 0 \cdot 3 - 1 \cdot 2 = -2$$

For a 3×3 Case, one *special case* we have is the **Basket-weaving Method**. The basket-weaving method of solving for determinants applies only to 3×3 matrices. It is the same process as getting the determinant of a 2×2 matrix, except we need to recopy the first and second columns of the matrix to the right of the original matrix, before doing the basket-weave method.

For example

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & 5 & -1 \\ -2 & 3 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 4 \\ 2 & 5 & -1 \\ -2 & 3 & -4 \end{pmatrix} \cdot \begin{bmatrix} 1 & -3 \\ 2 & 5 \\ -2 & 3 \end{bmatrix}$$

To do the basket weave method, we first add the diagonals going left to right

$$(1)(5)(-4) + (-3)(-1)(-2) + (4)(2)(3) = -2$$

Now we add the diagonal products going right to left

$$(-2)(5)(4) + (3)(-1)(1) + (-4)(2)(-3) = -19$$

Take the difference of the two sums to get the determinant

$$-2 - (-19) = 17$$

For matrices with a higher order, we can use the **Laplace Expansion**. The Laplace expansion is the basic technique for calculating determinants and applies to large matrices. To understand this, let us use an example of a general 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

1. Select any row or column in the matrix (say row 1, but preferably a row or column that has many 0s or 1s to simplify the process).

2. Calculate the *minor* for each element of the selected row/column

   - Since row 1 was selected, we can compute for 3 minors for each element in row 1, i.e. a_{11}, a_{12}, a_{13}.
   - The *minor* of a_{11}, denoted as $|M_{11}|$, is the determinant of the sub-matrix associated with a_{11}.
   - The sub-matrix associated with a_{11} is composed of the elements of the original matrix excluding the elements in the row and column which a_{11} appears. The submatrix associated with a_{11} is the following.

   $$a_{11} \text{ sub-matrix } = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \text{ taken from } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

   - Thus, the minor of a_{11} is $|M_{11}| = a_{22}a_{33} - a_{23}a_{32}$.
   - Do the same to get the minors of a_{12} and a_{13} where $|M_{12}| = a_{21}a_{33} - a_{23}a_{31}$ and $|M_{13}| = a_{21}a_{33} - a_{23}a_{31}$

3. Once you get the minors, you can now calculate the **cofactor** of each element of the selected row or column.

   - The cofactor of a_{ij} is its minor with an assigned algebraic sign.
   - The general formula for the cofactor is $|C_{ij}| = (-1)^{i+j}|M_{ij}|$.
   - In the forgoing illustration, the cofactors are:

   $$a_{11} : |C_{11}| = (-1)^{1+1}|M_{11}| = |M_{11}|$$

   $$a_{12} : |C_{12}| = (-1)^{1+2}|M_{12}| = -|M_{12}|$$

   $$a_{13} : |C_{13}| = (-1)^{1+3}|M_{13}| = |M_{12}|$$

4. Calculate the determinant using the following formula

   $$|A| = \sum a_{ij} \cdot |C_{ij}| = a_{11} \cdot |C_{11}| + a_{12} \cdot |C_{12}| + a_{13} \cdot |C_{13}|$$

For example, say you were given with Matrix A

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{pmatrix}$$

We select the row or column with the most zeros. In this case, let us select the first row.

$$|M_{11}| = (3)(6) - (5)(4) = -2$$
$$|M_{12}| = (2)(6) - (5)(3) = -3$$
$$|M_{13}| = (2)(4) - (3)(3) = -1$$

We then calculate for the cofactors

$$|C_{11}| = (-1)^{1+1}(-2) = -2$$
$$|C_{12}| = (-1)^{1+2}(-3) = 3$$
$$|C_{13}| = (-1)^{1+3}(-1) = -1$$

Lastly, we apply the formula for the determinant

$$|A| = (1)(-2) + (0)(3) + (4)(-1) = -6$$

Find the determinants of the following matrices

1. $\begin{pmatrix} 1 & -6 & 5 \\ 2 & 2 & 5 \\ -1 & -4 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 15 & 4 & 8 \\ -12 & -7 & 5 \\ 0 & -5 & 15 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 1 & 0 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 3 & 2 & 2 \\ 2 & 1 & 3 & 3 \end{pmatrix}$

4.3.4 Basic Properties of Determinants

There are a couple of properties and aspects of determinants we must take note of

1. The interchange of rows and columns does not affect the value of a determinant, i.e. $|A| = |A^T|$
2. The interchange of any 2 rows or any 2 columns will alter the sign but not the numerical value of the determinant
3. The multiplication of any 1 row or 1 column by a scalar k will change the value of the determinant k-fold
4. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the value of the determinant unaltered.
5. If at least 1 row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

6. A square matrix is singular if the determinant does not exist (i.e. is equal to zero). In other words, a square matrix is *singular* if one or more of its rows and columns are linearly dependent. On the contrary, a square matrix is *non-singular* if its determinant exists (i.e. is non-zero).

7. The determinant of a triangular matrix is simply the product of its diagonal entries.

4.3.5 Defining the Inverse of a Matrix

Theorem 4.3. *The inverse of a square matrix A is a square matrix B of the same order such that $AB = BA = I$ where I is an identity matrix*

For example, consider these two matrices

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1/2 \\ -1 & 1/2 \end{pmatrix}$$

If you multiply $A \cdot B$

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1/2 \\ -1 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly, if we do $B \cdot A$

$$\begin{pmatrix} 0 & 1/2 \\ -1 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, we can consider B as the inverse of A. From the definition, it is clear that if B is an inverse of A, then A is an inverse of B. However, there are such matrices that do not have inverses.

The inverse of a square matrix is unique. For if B and C are both inverses of A, then it must be the case that:

$$AB = I = BA, \qquad AC = I = CA$$

Then, it must also hold that

$$B = BI = B(AC) = (BA)C = IC = C$$

Since the inverse of a square matrix is unique, we identify it by denoting it as A^{-1}. Thus,

$$AA^{-1} = I = A^{-1}A$$

4.3.6 Solving for the Inverse of a Matrix

To get the inverse of a matrix, you must first compute for the determinant of the matrix. This implies that the *matrix must be both square and non-singular.*

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix}$

1. Derive the cofactor of *every* entry of A

2. Set up the matrix of cofactors

$$C = \begin{pmatrix} |C_{11}| & \dots & |C_{1m}| \\ \vdots & \ddots & \vdots \\ |C_{m1}| & \dots & |C_{mm}| \end{pmatrix}$$

3. Take the transpose of matrix C, which is also known as the **adjoint** of A, i.e. $C' = adj(A)$

4. Divide C' by the determinant of A (i.e divide the transpose of the matrix of cofactors by the determinant of A to get the inverse.

$$A^{-1} = \frac{C'}{|A|} = \frac{\begin{pmatrix} |C_{11}| & \dots & |C_{1m}| \\ \vdots & \ddots & \vdots \\ |C_{m1}| & \dots & |C_{mm}| \end{pmatrix}}{|A|} = \begin{pmatrix} \frac{|C_{11}|}{|A} & \dots & \frac{|C_{1m}|}{|A|} \\ \vdots & \ddots & \vdots \\ \frac{|C_{m1}|}{|A|} & \dots & \frac{|C_{mm}|}{|A|} \end{pmatrix}$$

As an example, find the inverse of the following matrix.

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{pmatrix}$$

We first derive the cofactor of every entry in A and set up the cofactor matrix. To get the cofactors, we have to first get the minors of every entry. The minors when computed are $|M_{11}| = -2, |M_{12}| = -3, |M_{13}| = -1, |M_{21}| = -16, |M_{22}| = -6, |M_{23}| = 4, |M_{31}| = -12, |M_{32}| = -3, |M_{33}| = 3$.

Therefore, the cofactors are just $|C_{11}| = -2, |C_{12}| = 3, |C_{13}| = -1, |C_{21}| = 16, |C_{22}| = -6, |C_{23}| = -4, |C_{31}| = -12, |C_{32}| = 3, |C_{33}| = 3$. This allows us to build the cofactor matrix.

$$C = \begin{pmatrix} -2 & 3 & -1 \\ 16 & -6 & -4 \\ -12 & 3 & 3 \end{pmatrix}$$

To obtain the adjoint, we simply take the transpose of the cofactor matrix, that is, $adj(A) = C'$

$$adj(A) = C' = \begin{pmatrix} -2 & 16 & -12 \\ 3 & -6 & 3 \\ -1 & -4 & 3 \end{pmatrix}$$

Finally, because the determinant of matrix A is -6 (which you can compute yourself), the inverse of matrix A is

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{\begin{pmatrix} -2 & 16 & -12 \\ 3 & -6 & 3 \\ -1 & -4 & 3 \end{pmatrix}}{-6} = \begin{pmatrix} 1/3 & -8/3 & 2 \\ -1/2 & 1 & -1/2 \\ 1/6 & 2/3 & -1/2 \end{pmatrix}$$

Find the inverses of the following matrices

1. $\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 4 & 1 & -1 \\ 0 & 3 & 3 \\ 3 & 0 & 7 \end{pmatrix}$

3. $\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{pmatrix}$

5 Course Problem Sets

All of your grade for the course is hinged on the course problem sets. Each problem set constitutes an equal proportion of your final grade. The deadline for any and all problem sets is on *Week 14* of the Term. However, it is recommended that students submit the problem sets as early as they can to avoid cramming. Try and space out your work flow by answering the problem set after the lecture or during your fourth hour. All problem sets may be done *computerized* using MS Word with Equation Editor/MathType, Pages, or LaTeX. Alternatively, you may opt to answer all problems manually and scan your answers. Please ensure that your handwritting is *readable* if this is the alternative you would prefer. All problem sets are to be submitted to their respective Canvas assignment entries on or before the Friday of the 14th week of the term. As a safety net, you can also submit your problem set to justin.eloriaga@dlsu.edu.ph. All problem sets should be submitted in PDF form.

In terms of grading, your score will be in the grading scheme of DLSU, that is; 4.0, 3.5, 3.0, 2.5, 2.0, 1.5, 1.0, and 0.0. To obtain your final grade, the average of the seven (7) highest problem sets will be computed for. The problem set with the lowest score shall be discarded. Note that the rounding off points will be discussed to you in class.

FREQUENTLY ASKED QUESTIONS

1. **Q: Will you accept answers without any solutions?**
   A: No! We will strictly follow a no-solution, no-credit policy
2. **Q: Can I collaborate with my fellow classmates?**
   A: Yes! But if there is any evidence of copy-paste answers, such as directly copying the solutions, final answer, and reasoning of your classmate, the grade will be divided by the suspected number of copies. You are still in graduate school, try and accomplish these things with integrity.
3. **Q: If you will only take the highest 7 problem sets, can you opt to just submit 7 problem sets?**
   A: Yes! But historically, this was not an effective strategy
4. **Q: Are all problem sets the same in difficulty on the average?**
   A: No! There are some that are harder than others and some that are relatively easy.
5. **Q: What are your consultation hours?**
   A: Thursday 19:30 - 20:30. Please set any consultation through my email at least 36 hours in advance.
6. **Q: Do you have any recommended schedule for submission of Problem Sets?**
   A: Yes! Please find the table below.

Problem Set	Recommended Submission
1	Week 3
2	Week 5
3	Week 6
4	Week 8
5	Week 9
6	Week 11
7	Week 12
8	Week 14

5.1 Problem Set 1: Single Variable Differentiation

1. Using the difference quotient, find the derivative of the following functions

   - $f(x) = 120/x$
   - $f(x) = 90x^4 + 60x^2$
   - $f(x) = \sqrt{x}$

2. Find the derivative or y'

   - $y = \frac{x^3+2x+1}{x^2+3}$
   - $y = \frac{x^e}{e^x}$
   - $y = 2 - x^4 e^{-x}$
   - $y = \left(\frac{x}{1+x}\right)^5$
   - $y = \frac{2x^{-1}-x^{-2}}{3x^{-1}-4x^{-2}}$
   - $y = \alpha^{\beta x^2}$
   - $y = (2x^3 + 5)^{3/2}$
   - $y = \sqrt[6]{x^5} + \sqrt[4]{x^3} + \sqrt{x^5 + 2}$
   - $y = \frac{1}{x^2} - \frac{4}{2^{x^2}}$
   - $y = \frac{1}{\ln(\ln(x))-1}$
   - $y = e^{e^x}$
   - $y = x^4 e^{-2x}$
   - $y = \sqrt{\sqrt{\sqrt{\sqrt{x^2 - 1}}}}$
   - $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
   - $y = \frac{1-\sqrt{1+x}}{x}$
   - $y = \frac{1}{(x^2+x+1)^5}$

3. Find the first and second-order derivatives of the following functions

   - $y = 2x - 5$
   - $y = \frac{1}{3}x^9$
   - $y = e^x \ln(x)$
   - $y = \ln(\ln(x^2))$
   - $y = (1 + x^2)^1 0$
   - $y = \frac{x^4}{4} + \frac{x^3}{3} + \frac{5^2}{2}$
   - $y = \frac{1}{2}(e^x - e^{-x})$

4. Solve for $f'(x)$ or $\frac{dy}{dx}$ using implicit differentiation

   - $x^3 + 3x^2 y - 6xy^2 + 2y^3 = 0$
   - $x^3 + y^3 = 4$
   - $e^{xy} = e^{4x} - e^{5y}$
   - $x^y + y^4 = 4 + 2x$
   - $x = 3 + \sqrt{x^2 + y^2}$

5. A manufacturer's cost function is given by $C(x) = 0.5x^3 - 10x^2 + 110x$, where x denotes the output level. What is the marginal cost when output is 10? Interpret your answer, then verify by computing for the actual cost of producing the 11th item.

6. The total revenue obtained from the sale of x cellphones is given by $R(x) = 10x - 0.02x^2$, where $R(x)$ denotes revenue and x denotes the quantity sold. Find the marginal revenue when 100 cellphones are sold, then interpret your result.

7. A study in transport economics uses the relation $T = 0.4K^{1.06}$, where K is expenditure on building roads and T is a measure of traffic volume. Find the elasticity of T with respect to K, then interpret the result.

8. The demand for a commodity is given by $Q = 100 - 8P$, where P denotes the price and Q denotes the quantity demanded.

   - Find the price elasticity of demand function.
   - What is the price elasticity of demand when $P = 5$? Is the demand price elastic, unit elastic, or inelastic? Interpret the result.

9. The inverse demand function for a commodity is given by $P = 100 - 0.025Q$, where P denotes the price and Q denotes the quantity demanded.

   - Find the total revenue function.
   - Find the marginal revenue when $Q = 100$. Interpret the result.

5.2 Problem Set 2: Multivariate Differentiation

1. Find the first-order partial derivatives with respect to all variables of the following functions.
   - $f(x,y,z) = z^3 - 3x^2y + 6xyz$
   - $f(x,y,z) = \frac{xyz}{x+y+z}$
   - $f(x,y) = e^{xy}\ln(xy)$
   - $f(x,y) = e^{x^2+xy}$
   - $f(x,y) = x^3e^{y^2}$
   - $f(x,y) = \frac{x^2}{y} + \frac{y^2}{x}$
2. Find the first and second order (both cross and direct) of the following functions. Check to see if Young's Theorem is satisfied.
   - $f(x,y) = x^7 - y^7$
   - $f(x,y) = x^5\ln(y)$
   - $f(x,y) = (x^2 - 2y^2)^5$
   - $f(x,y) = 5x^4y^2 - 2xy^5$
   - $f(x,y) = \frac{x^2-4y^3}{2xy}$
   - $f(x,y) = \frac{x-y}{x+y}$
   - $f(x,y) = x/y$
   - $f(x,y) = xy^2 - e^{xy}$
3. Prove that if $z = (ax+by)^2$, then $xf_x + yf_y = 2z$.
4. Let $z = \frac{1}{2}\ln(x^2+y^2)$. Show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$
5. Find the total differential of the following functions.
   - $f(x,y) = 3x^3(8x - 7y)$
   - $m(x,y) = \frac{2xy}{x+y}$
   - $z = (x-3y)^3$
   - $u(x,y) = -5x^3 - 12xy - 6y^5$
   - $f(x,y) = 7x^2y^3$
   - $z = e^{x^2-y^2}$
   - $h(x,y) = x^3y + x^2y^2 + xy^3$
   - $f(x,y) = \sqrt{9 - x^2 - y^2}$
6. Find both the linear and quadratic Taylor approximations of the following functions:
   - $f(x) = xe^{2x}$ about $x_0 = 0$
   - $f(x) = AK^\alpha$ about $K_0 = 0$
   - $f(x) = 5(\ln(1+x) - \sqrt{1+x})$ about $x_0 = 0$
7. Identify, if any, the local maxima and local minima of the given functions using either the first-derivative test or the second derivative test. In addition, identify the inflection point/s, if any, and determine the regions of concavity and convexity.
   - $f(x) = \frac{1}{3}x^3 - x^2 + x + 10$
   - $m(x) = -2x^2 + 8x + 25$
   - $z(x) = \frac{x^2}{x^2+2}$
   - $u(x) = 3x^5 - 5x^3$

5.3 Problem Set 3: Economic Applications of Differentiation

1. Consider a typical consumer whose preferences can be represented by a Constant Elasticity of Substitution (CES) utility function $U(x_1, x_2) = (x_1^\rho + x_2^\rho)^{-1/\rho}$ where x_1 represents the amount of good 1 consumed and x_2 represents the amount of good 2 consumed.

   - Take the total differential of $U(x_1, x_x)$
   - Prove that the consumer is non-satiated. (Hint: Obtain the marginal utility of each good and prove that this is positive while stating the relevant domain).
   - Prove the phenomenon of diminishing marginal utility, that is, the additional utility per additional consumption will be lower for every additional consumption unit thereafter.
   - Prove that given the relevant domain, the consumer will never be "less" happy with an additional consumption of any good.
   - Demonstrate that Young's Theorem holds
   - Do the first five bullets but using the function $U(x_1, x_2) = (x_1 - \beta_1)^{\alpha_1}(x_2 - \beta_2)^{\alpha_2}$ where $\beta > 0$ is some subsistence parameter and $0 \leq \alpha \leq 1$ is some parameter.
   - Do the first five bullets but using the function $U(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ where $0 \leq \alpha \leq 1$ is some parameter.

2. A firm has a cost function $C = \frac{1}{3}Q^3 - 7Q^2 + 111Q + 50$ and a demand function $Q = 100 - P$.

   - Write out the total revenue function TR of the firm as some function of Q
   - Formulate the profit function π as a function of Q.
   - Find the profit-maximizing level of output Q^*
   - How much is the maximum profit?

3. A firm's production function is $Q = 12L^2 - (1/20)L^3$, where L denotes the number of workers.

   - What size of the workforce (L^*) maximizes the output Q?
   - What size of the workforce maximizes output per worker Q/L?

4. A firm produces $Q = 2\sqrt{L}$ units of a commodity when L units of labor are employed. If the price obtained per unit is 160 pesos, and the price per unit of labor is 40 pesos, what value of L maximizes profits?

5. Let $R(Q) = 80Q$ and $C(Q) = Q^2 + 10Q + 90)$. The firm can at most produce 50 units.

   - How many units must be produced for the firm to make a profit?
   - How many units must be produced for the firm to maximize profits?

6. Mr. Constantine wishes to mark out a rectangular flower bed using a wall of his house as one side of the rectangle. The other three sides are to be marked by wire netting, of which he has only 64 ft. available. What are the length L and width W of the rectangle that would give him the largest possible planting area? Check to make sure that your answer gives you the largest, not the smallest area.

5.4 Problem Set 4: Techniques of Integration

1. Evaluate the following indefinite integrals using whatever technique is feasible

   - $\int (20x^2 + 2x^{-1} - 3x^{5/3})$ dx
   - $\int \frac{1}{t(\ln(t))^2}$ dt
   - $\int r(\ln r)^2$ dr
   - $\int \frac{x^3}{(2-x^2)^{5/2}}$ dx
   - $\int \frac{18x^2+12}{4x^3+8x}$ dx
   - $\int x^3 e^{x^2}$ dx
   - $\int 5xe^{5x^2+3}$ dx
   - $\int \frac{x}{x^2+5x+6}$ dx
   - $\int \frac{x^2+2}{x(x+2)(x-1)}$ dx

2. Evaluate the following definite integrals using whatever technique is feasible

   - $\int_0^2 \frac{1}{(3+5x)^2}$ dx
   - $\int_0^{3\ln 3} (e^{x/3} - 3)$ dx
   - $\int_0^{10} (1 + 0.4t)e^{-0.05t}$ dt
   - $\int_0^2 (x-2)e^{-x/2}$ dx
   - $\int_a^b (px + q)$ dx
   - $\int_{-1}^1 \frac{1}{2}(e^x + e^{-x})$ dx
   - $\int_0^1 \frac{x}{\sqrt{1-x^2}}$ dx

3. Evaluate the following improper integrals using whatever technique is feasible and determine whether it is convergent or divergent

   - $\int_{-\infty}^{\infty} xe^{-cx^2}$ dx
   - $\int_0^{\infty} \frac{e^x}{(1+e^x)^2}$ dx
   - $\int_{1/3}^3 \frac{1}{\sqrt[3]{3x-1}}$ dt
   - $\int_e^{\infty} \frac{1}{x(\ln x)^2}$ dx
   - $\int_0^a \frac{x}{\sqrt{a^2-x^2}}$ dx
   - $\int_1^{\infty} 3x^2 e^{-x^3}$ dx
   - $\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}$ dx

4. Say a firm has a marginal cost function of $MC(x) = 0.003x^2 - 0.08x + 100$ where x represents the quantity produced. Find the total cost function.

5. Suppose a firm has a marginal revenue function equal to $MR(x) = 500 - 0.1x$ where x represents the quantity produced. Find the Total Revenue function of the firm. From there, calculate the demand function.

6. Consider a demand function $P = 200 - 0.2Q$ and a supply function of $P = 20 + 0.1Q$.

   - Compute for the equilibrium price and quantity.
   - Calculate for the social surplus or *net market surplus*

7. Consider a demand function $Q = 500 - 50P$ and a supply function of $Q = 50 + 25P$.

    - Compute for the equilibrium price and quantity.
    - Calculate for the consumer surplus
    - Calculate for the producer surplus
    - Calculate for the social surplus

5.5 Problem Set 5: Deep Dive on Matrix Algebra

1. Identify whether the following statements are "True" or "False"

   - Matrix addition, subtraction, and multiplication all satisfy the associative property.
   - Matrix multiplication is not commutative, although matrix addition is commutative.
   - Non-singular matrices have no inverses
   - Idempotent matrices may not necessarily be square matrices.
   - For any square matrix, it is always the case that $|AB| = |A||B|$.

2. Given the following matrices, perform the following operations

$$A = \begin{pmatrix} 6 & -3 & -1 \\ -4 & 3 & 8 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 5 \\ 1 & 4 & -1 \end{pmatrix}, C = \begin{pmatrix} 3 & 0 \\ 1 & -2 \\ -1 & 2 \end{pmatrix}, D = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$$

   - CA
   - BCD
   - $3AC - 2D$
   - $A + B$
   - $B^T D$
   - $D^T A$

3. Let $A = \begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$. For what values of t does A have an inverse?

4. Prove that the matrix $A = Ix(x^T x)^{-1} x^T$ is an idempotent matrix. Do both x and $(x^T x)$ have to be square matrices for the condition to be satisfied?

5. Compute for the determinant and the inverse of the following matrix

$$A = \begin{pmatrix} 2 & 6 & 1 & 7 \\ 1 & 0 & 1 & 2 \\ 4 & 5 & 9 & 8 \\ 3 & 0 & 2 & 1 \end{pmatrix}$$

6. Given the following system of equations:

$$x_1 - 8x_2 + x_3 = 4$$
$$-x_1 + 2x_2 + x_3 = 2$$
$$x_1 - x_2 + 2x_3 = -1$$

   - Express the system of linear equations in matrix format
   - Find the determinant of your coefficients matrix
   - Solve for the solution values via matrix inversion
   - Solve for the solution values using Cramer's Rule

7. Use matrix inversion to solve for the unknowns in the system of linear equations given below. In addition, prove that you get the same answer using Cramer's rule.

$$2x_1 + 4x_2 - 3x_3 = 12$$
$$3x_1 - 5x_2 + 2x_3 = 13$$
$$-x_1 + 3x_2 + 2x_3 = 17$$

5.6 Problem Set 6: Economic Applications of Matrix Algebra

1. The equilibrium conditions for two related markets (siopao and hakaw) are given by

$$18p_h - p_s = 87$$
$$-2p_h + 36p_s = 98$$

   Find the equilibrium price for each market using matrix inversion.

2. Given the IS equation $0.3Y + 100i - 252 = 0$ and the LM equation $0.25Y - 200i - 176 = 0$. Find the equilibrium level of income and rate of interest.

3. Given $Y = C + I_0$ where $C = C_0 + bY$, use matrix inversion to find the equilibrium level of Y and C. Analyze these equilibrium forms and give some economic intuition.

5.7 Problem Set 7: Unconstrained Optimization

1. Identify the local maxima and/or minima. Verify using either the first- or second-order derivative test.

   - $f(x) = e^x/x$
   - $f(x) = 2x^3 - 5x_4^2x - 7$
   - $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$
   - $f(x) = x - 2\ln(x+1)$

2. Consider the total cost function given by $TC(Q) = 5Q^3 - 10Q^2 + 16Q$

   - Solve for the level of Q that will minimize *average cost*. Verify your answer using the first-derivative and second-derivative test.
   - Using your first-derivative test results in the first bullet, at which range of output is the slope of the average function positive? At which range of output is this slope of the average cost function negative?
   - Compute for the minimum average cost

3. Given the total cost function $TC = 0.5^q - 10q^2 + 80q$

   - Find the average cost AC and the marginal cost MC
   - At what level of output is average cost at a minimum? Verify using the second-order derivative test.
   - At what level of output is marginal cost at a minimum? Verify using the second-order derivative test.

4. Suppose that a monopolist has the following demand function and cost function

$$P(Q) = 10 - \frac{Q}{1000}, \quad C(Q) = 5000 + 2Q$$

   Find the output level that maximizes profits, verify whether your answer is indeed a maximum, and find the maximum profit.

5. Suppose that a firm has a revenue function given as $R(x, y) = \rho x + qy$ and a total cost function given as $C(x, y) = \alpha x^2 + \beta y^2$

   - What are the needed first order conditions for profit maximization?
   - What is the optimal x^* and y^*?
   - What is the maximum profit pi^*?
   - Verify that $\frac{\partial \pi^*}{\partial \rho} = x^*$ and $\frac{\partial \pi^*}{\partial q} = y^*$

6. Consider a firm that produces two products Q_1 and Q_2 at two different prices P_1 and P_2. The cost function is given by $C(Q) = 4Q_1^2 + 2Q_1Q_2 + 4Q_2^2$.

   - Find the profit function
   - What are the optimal quantities of Q_1 and Q_2 that will maximize profit?
   - Verify that the optimal quantities will indeed yield a maximum profit.

7. A competitive firm produces output given by the production function $Q = L^{0.25}K^{0.25}$, where$L$ and K are labor and capital respectively. The firm charges a price of 4 PHP per unit of output produced, and faces a price of 2 PHP per unit of labor and a price of 4 PHP per unit of capital employed.

- Compute for the profit-maximizing levels of labor and capital that the firm should employ.
- Verify that the values of L and K obtained in the first bullet are indeed profit-maximizing.
- Compute for the maximum profit

8. Suppose demand for good 1 is given by $p_1 = 26 - \frac{3q_1}{2}$ and demand for good 2 is given by $p_2 = 72 - 2q_2$. The firm has the cost function $C = 120 + q^2$ for both goods where $q = q_1 + q_2$. Compute the following

   - The profit maximizing level of output for each good, i.e., q_1 and q_2.
   - Verification that the quantities obtained in the first bullet will indeed maximize profit
   - The prices p_1 and p_2 that the firm will charge for each good to maximize profit
   - The maximum profit

5.8 Problem Set 8: Constrained Optimization

1. A competitive firm charges a price of 50 PHP per output produced, given by $Q = L^{0.25}K^{0.25}$. The input costs of labor and capital are 2 PHP and 2 PHP respectively. However, the firm has a required input capacity of only $L + K = 20$.
   - Compute for the profit-maximizing levels of labor and capital using the Lagrangean method
   - Verify that the second-order sufficient condition for a local maxima is met
   - Compute for the maximum profit

2. Your preferences are represented by the utility function $U = (x_1^{0.5} + x_2^{0.5})^2$, where the price of good 1 is 4 PHP and the price of good 2 is 3 PHP. Your income is given by 50 PHP.
   - Set up the Lagrangean function for this utility-maximization problem with constraint.
   - Compute for the utility-maximizing quantities of x_1 and x_2.
   - What is the maximum level of utility you can attain given your utility function and budget constraint?

3. Suppose you are trying to minimize your expenditures $E = 4x_1 + 3x_2$ subject to the target maximum utility level $(x_1^{0.5} + x_2^{0.5})^2 = \frac{175}{6}$.
   - Set up the Lagrangean function for this expenditure-minimization problem with constraint
   - Compute for the expenditure minimizing quantities of x_1 and x_2.
   - What is the minimum expenditure you can incur given the level of utility you have to attain?

4. Chino has a quasi-linear utility function given by $U(x_1, x_2) = x_1 + 6x_2^{\frac{2}{3}}$, where prices of goods 1 and 2 are given by p_1 and p_2, respectively. If Chino's budget is given by m,
   - Solve for the utility-maximizing quantities of x_1, x_2, and λ
   - Find the maximum utility.
   - Show that the change in maximum utility due to a peso change in income m is equal to the Lagrange multiplier you got in the first bullet.

5. A firm pays a price w for each unit it employs of its labor x. It also incurs fixed costs equal to F. If the firm receives a price p for each unit of output produced, and it produces output given by $f(x) = 4x^{1/4}$, how much of labor x should it employ to maximize profits? Verify whether your answer is indeed a maximum.

6. A firm has three factories each producing the same item. Let x, y and z denote the respective output quantities that the three factories produce to fulfill an order of 2000 units in total. The cost functions for the three factories are the following:

$$C_1(x) = 200 + \frac{1}{100}x^2$$
$$C_2(y) = 200 + y + \frac{1}{300}y^2 + y$$
$$C_3(z) = 200 + 10z$$

   - Find the values of x, y and z that minimize the total cost C.
   - Verify that the solutions yield minimum costs.
   - What is the minimum cost?

7. A single-product firm intends to produce 30 units of output as *cheaply* as possible. By using K units of capital and L units of labor, it can produce $\sqrt{K} + L$ units. Suppose that the prices of labor are 1 and 20, respectively.

   - Form the Lagrangean to solve for a minimum cost
   - Find the optimal choices of K and L
   - What is the associated cost?
   - What is the additional cost of producing 31 units rather than 30 units?

6 Course Syllabus

ECO501M: Mathematical Economics for Master in Applied Economics

Instructor: Justin Raymond S. Eloriaga
Class schedule: Saturday 09:00 - 12:00 (Fully Online) Term 1, A.Y. 2020 - 2021
Consultation: By email (justin.eloriaga@dlsu.edu.ph)

6.1 Course Description

This course serves as an introductory course in mathematics for economic analysis at the graduate level. The course focuses on the mathematical foundations used in economic theory, and the objective is for students to learn how to use the necessary mathematical tools in studying and understanding economics. The course discusses concepts on the applications of differential calculus and integral calculus, linear and non-linear optimization, and matrix algebra. At this level, it is important that students should be able to successfully complete all of the calculations needed with consistency and accuracy, and consequently, develop the ability to interpret and understand mathematical equations and calculations. After building on students' mathematical foundations, the course shifts over to economic applications and analyses. At this point, mathematical theories with economic applications will be covered in class to help students use the language of mathematics to describe and analyze economic models and solve economic problems.

School of Economics' Course Learning Outcomes:

Knowledge	· Apply both qualitative and quantitative concepts of the derivative of a function. · Interpret the concept of a definite integral as the area of a given region. · Differentiate differential and integral calculus and the relationship between them.
Skills	· Correctly apply differentiation rules. · Apply differential calculus in an economic context. · Demonstrate the applicability of integral calculus in the capital accumulation and welfare concept of economics. · Solve problems of integration using the different techniques of integral calculus. · Solve matrix algebra problems and apply the concepts in economic applications such as input-output models · Have a full grasp of both linear and non-linear optimization techniques · Confidently express graphical and conceptual models in equation form.
Behavior	· Exhibit resilience in solving economic problems mathematically. · Exhibit willingness to work well within a team, to be open-minded and receptive to others' insights and constructive feedback, and to develop initiative

6.2 Main References

6.2.1 Required References

- Chiang, A. and K. Wainwright. (2005). Fundamental Methods of Mathematical Economics. 4th edition. McGraw-Hill/Irwin: New York. (Main Reference)
- Sydsæter, K. and P. Hammond. (2012). Essential Mathematics for Economic Analysis, 4th edition. Pearson Education Limited: England
- Eloriaga, J. (2020). Mathematical Economics

6.2.2 Other References

- Danao, R. (2011). Mathematical Methods in Economics and Business. The University of the Philippines Press: Quezon City.
- Danao, R. (2017). Core Concepts of Calculus with Applications. The University of the Philippines Press: Quezon City.

6.3 Grading System and Requirements

Given the unique circumstance of the term, the subject shall no longer require students to undertake a summative midterm or final examination. Instead, the course material shall be broken down into eight (8) problem sets divided equally. These 8 problem sets are *long* problem sets which cover all grounds on the topics planned to be covered. Students are required to accomplish at least 7 problem sets as the average of the highest 7 problem sets will be the student's corresponding grade. All problem sets are *individually accomplished* but collaboration (to the extent of discussion) is certainly permitted.

Grading Component		$96 \leq grade \leq 100$	4.0
Long Problem Sets (8 in Total)	100%	$90 \leq grade \leq 95.9\ldots$	3.5
		$84 \leq grade \leq 89.9\ldots$	3.0
		$78 \leq grade \leq 83.9\ldots$	2.5
		$72 \leq grade \leq 77.9\ldots$	2.0
		$66 \leq grade \leq 71.9\ldots$	1.5
Total	100%	$60 \leq grade \leq 65.9\ldots$	1.0
NOTE: Passing Mark	72%	grade <60	0.0

6.4 Lectures

The course will revolve heavily on the prepared content in the Mathematical Economics lecture notes by Justin Eloriaga. The class will operate around the content of the material and examples contained inside the lecture notes shall be answered in class. Students are encouraged to immediately work on the problem sets thereafter to have a consistent flow. In addition, lecture videos on YouTube (Channel Name: *Justin Eloriaga*) concerning various topics in Mathematical Economics are also there for reference. With the class' consent, all synchronous classes will be recorded and uploaded to an *unlisted*[1] playlist on the same channel as well.

During synchronous lectures, it is highly encouraged that students' webcam are turned on while mics are turned off. No penalty will be incurred from the negligence to follow the stated policy. Attendance will not be monitored due to conditions arising from the pandemic.

[1] Only people with the video link may find and view the video

6.5 Course Plan

Topics	**Activities**
Course Introduction (Week 1) **Introductory Concepts on Differentiation** The Difference Quotient The Derivative Single Variable Differentiation	 Lecture Class Exercise Problem Sets
Further Dive on Differentiation (Week 2-3) Multiple Variable Differentiation Single Variable Optimization Taylor Approximations Implicit Differentiation Fermat's Theorem Concavity and Convexity Rolle's Theorem Weirstrauss' Theorem	 Lecture Class Exercise Problem Sets
Introductory Integration (Week 4) The Anti-derivative Indefinite Integration The Area Under a Curve The Definite Integral The Fundamental Theorem of Calculus	 Lecture Class Exercise Problem Sets
Techniques of Integration (Week 5-6) Integration by Substitution Integration by Parts Integration by Partial Fractions Integration by Partial Fractions with Repeated Factors Improper Integrals Area Between Curves Economic Applications of Integration	 Lecture Class Exercise Problem Sets
Introduction to Matrix Algebra (Weeks 7-8) Introduction to Matrices Matrix Operations Solving Systems of Linear Equations using Matrices Determinants Cramer's Rule and Matrix Inversion	 Lecture Class Exercise Problem Sets
Deep Dive on Matrix Algebra (Week 9) Eigenvalues and Eigendecomposition Comparative Statics Economic Applications of Matrix Algebra	 Lecture Class Exercise Problem Sets
Unconstrained and Constrained Optimization (Weeks 10-11) The Extremum Solving for Local Maxima and Local Minima The Hessian Matrix The Lagrangean Method The Bordered Hessian Matrix Linear Programming The Lagrangean Method Duality Conditions	 Lecture Class Exercise Problem Sets
Individual Consultation for Select Topics (Weeks 12-13)	
Submission of Problem Sets (Week 14)	

6.6 Course Guidelines

1. We shall utilize AnimoSpace as the virtual classroom for the course. All announcements and classroom materials shall be posted in the Canvas website, which you can access via dlsu.instructure.com. You'll have to use your DLSU email to log in.
2. We shall devote at most 3 hours of synchronous learning sessions each week, which will cover lectures, class exercises, and live online consultations and discussions. At most 3 hours per week shall be allocated to asynchronous learning that shall cover the time for problem sets and consultations which can be done through AnimoSpace or social media such as Zoom or Facebook.
3. Synchronous learning sessions will be saved for the benefit of the students who cannot be online on schedule. In addition, short video clips of the topics and concepts covered in each chapter/module shall be available for the students to catch up. These short videos can be accessed through the instructor's YouTube page.
4. As noted above, occasional exercises will be done during synchronous (i.e. conference) learning sessions. The problems are typically included in the slides. Students are encouraged to collaborate with the rest of the class and ask questions during the session.
5. Problem sets will be assigned after each topic. Asynchronous channels are also available for students in case they need assistance with regards to the lessons. The students are also encouraged to collaborate with their classmates in answering the problem sets.

6.7 Contact and Consultation Hours

My consultation hours are from 18:00 - 19:00 (Thursday) over Zoom. Please set an appointment at least 24 hours in advance. Consultation is strictly by appointment only. All contact may be made through justin.eloriaga@dlsu.edu.ph or through 09260321823. Alternatively, students may fill up the contact form in justineloriaga.com

Syllabus prepared by:

Justin Raymond S. Eloriaga

Noted by:

Dr. Arlene B. Inocencio
Department Chair

Dr. Marites M. Tiongco
Dean

About Your Professor

Apple Teacher

Justin Raymond S. Eloriaga is a lecturer at the De La Salle University School of Economics and a Central Bank Associate at the Bangko Sentral ng Pilipinas. He has taught various courses in both the undegraduate and graduate school with a primary focus on advanced econometrics and microeconomic theory. He obtained his Bachelor of Science in *Applied Economics* from De La Salle University and graduated *summa cum laude*. He also obtained a Master of Science in *Economics* from the same university, completing the rigorous ladderized track in the fastest time recorded in the school's history. He was also the 17th Commissioner of the Young Economist's Convention of the Economics Organization and a founding executive board member of the Lasallian Graduate Economics Society. He was also one of the top young economists of 2019 as inducted by the Philippine Economics Society.

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